PRE-CALCULUS 12

Seminar Notes

Learning Guides 1 & 2

TRANSLATIONS & TRANSFORMATIONS

A **transformation** is an operation which moves (or maps) a figure from an original position to a new position. Transformations we will consider are translations, reflections, expansions and compressions, and reciprocal transformations.

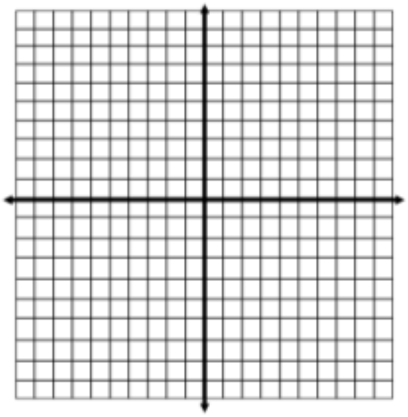
To get a full understanding of the transformations of various functions we must be able to visualize the basic functions before any moving…

"the original position."

The Original Position of Functions

The Linear Function:

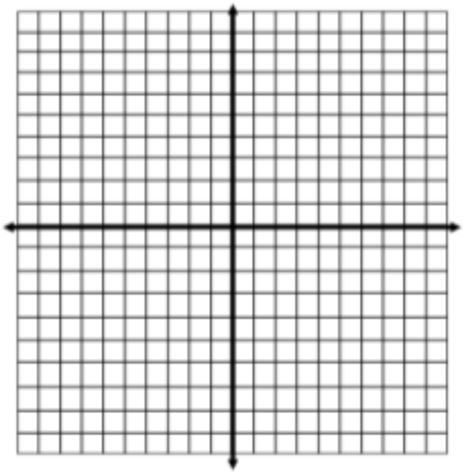
Basic Equation: $y = x$



a) Slope-Intercept Form:
b) Slope:
c) Domain, Range:
d) Equation of a vertical line:
e) Equation of a horizontal line:

The Quadratic Function:

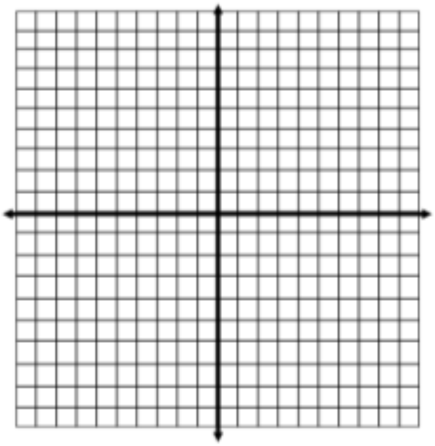
Basic Equation: $y = x^2$



a) Standard Form:
b) Domain, Range:
c) Vertex:
d) Equation of Axis of Symmetry:

The Square Root Function:

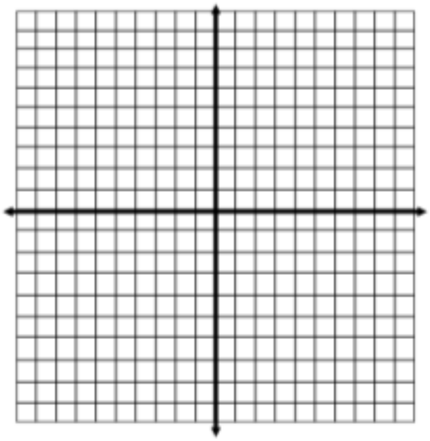
Basic Equation: $y = \sqrt{x}$



a) Standard Form:
b) Vertex:
c) Domain, Range:

The Absolute Value Function:

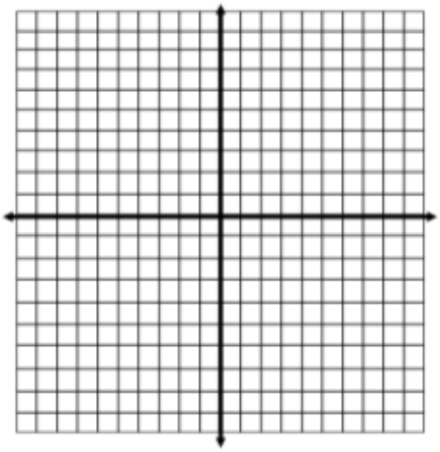
Basic Equation: $y = |x|$



a) Standard Form:
b) Vertex:
c) Domain, Range:

The Cubic Function:

Basic Equation: $y = x^3$

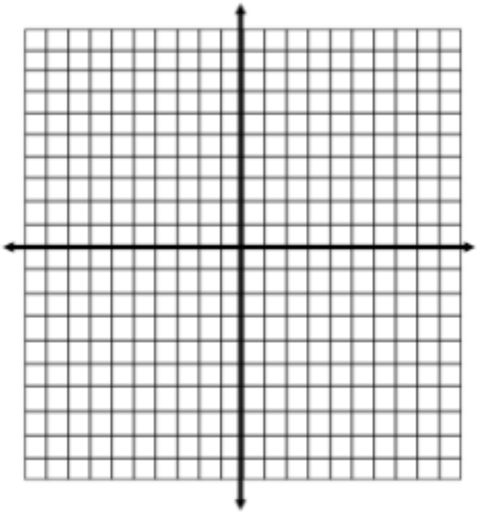


a) Standard Form:
b) Centre:
c) Domain, Range:

The Reciprocal Function:

Remember Brackets
&
Label Asympt.

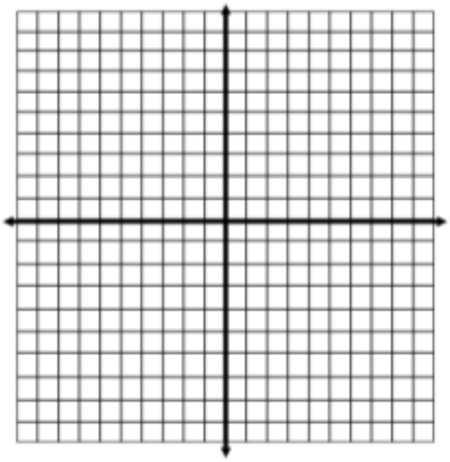
Basic Equation: $y = \frac{1}{x}$



a) Standard Form:
b) Domain, Range:
c) Asymptotes:

The Exponential Function:

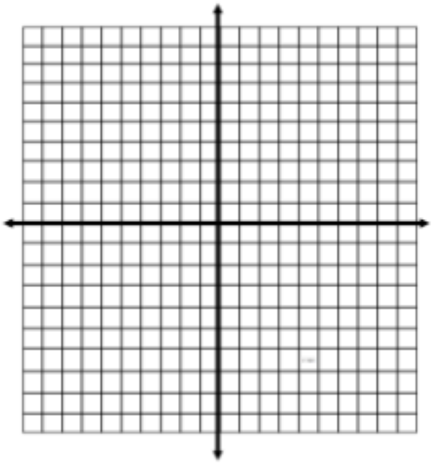
Basic Equation: $y = 2^x$



a) Standard Form:
b) Domain, Range:
c) Asymptotes:

The Logarithmic Function:

Basic Equation $y = \log_2 x$



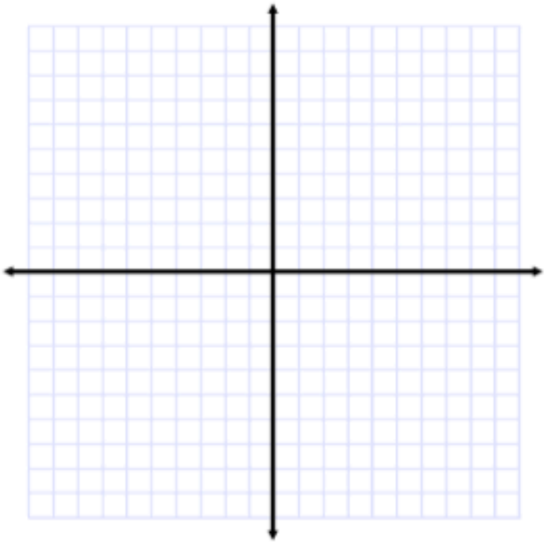
a) Standard Form:
b) Domain, Range:
c) Asymptotes:

Topic 1 Horizontal & Vertical Translations

Example 1 - **Comparing** y = f(x) **to** y = f(x – h)

Clear all functions. Enter $y_1 = |x|$ $y_2 = |x - 2|$ $y_3 = |x + 1|$

Sketch graphs.



What happens to the function?

TRY: Give the new equation for each translation on the square root function $y = \sqrt{x}$

a. 4 units right

b. 3 units left

Complete the sentences.

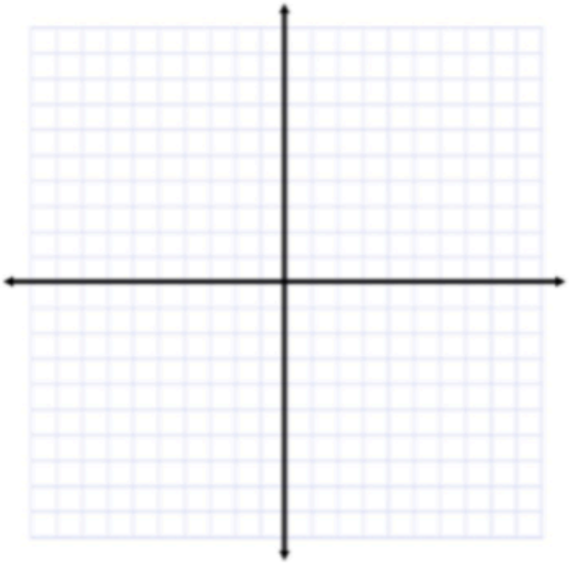
If y = f(x + h) the graph shifts ...

If y = f(x – h) the graph shifts ...

Example 1b - **Comparing** $y = f(x)$ **to** $y = f(x) + k$ **or** $y - k = f(x)$

Clear all functions. Enter $y_1 = x^2$ $y_2 = x^2 + 4$ $y_3 = x^2 - 2$

Sketch graphs.



What happens to the function?

TRY: Give the new equation for each translation on the quadratic function $y = x^2$

a. 4 units up

b. 3 units down

Complete the sentences

If $y = f(x) + k$ the graph shifts …

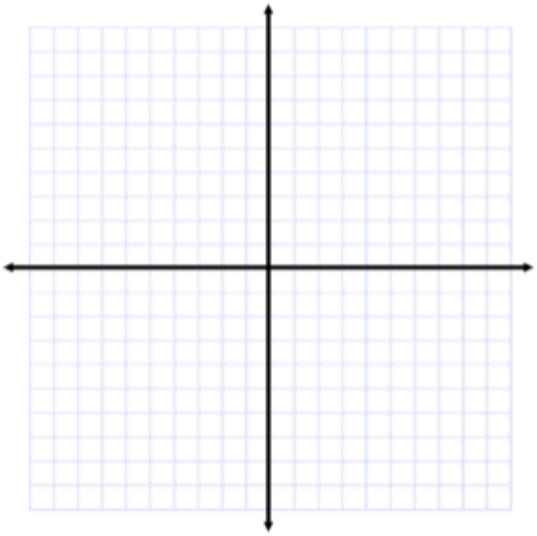
If $y = f(x) - k$ the graph shifts …

Example 2 - Horizontal & Vertical Translations

Sketch the function $f(x) = (x - 3)^2 - 1$

What is the base equation?

What are the translations?

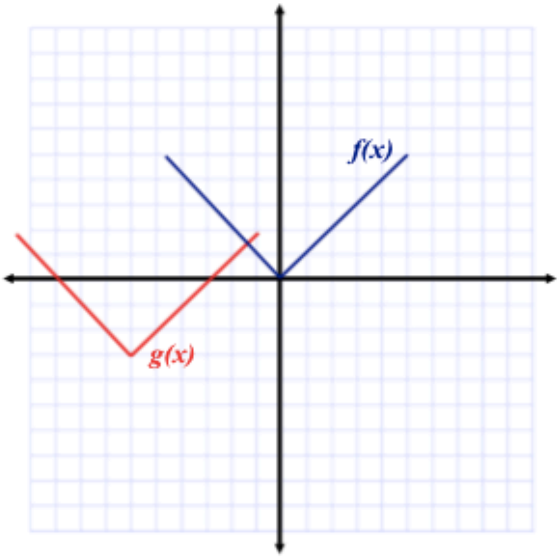


a) What is the domain and range of this new function?

b) What are the x- and y-intercepts?

Example 3 - Determine the Equation of a Translated Function

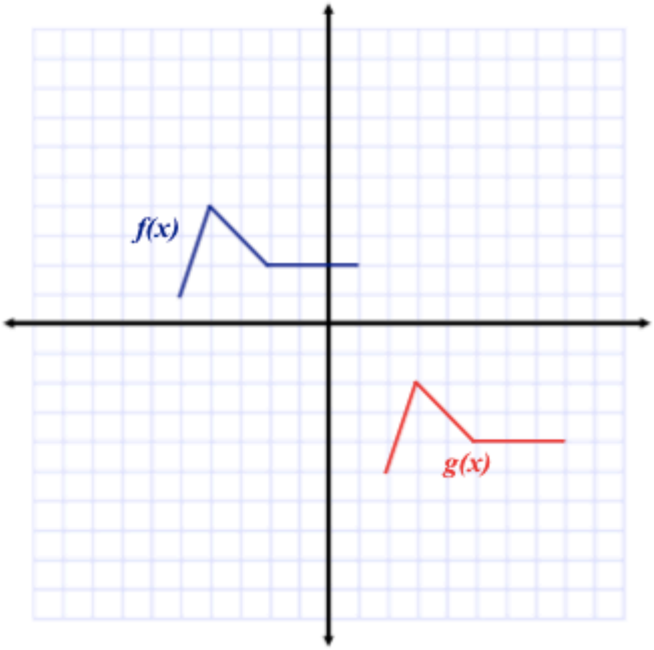
Determine the translation that has been applied to the graph $f(x)$ to obtain the graph of $g(x)$. Determine the equation of the translated function in the form $y - k = f(x - h)$ ***or*** $y = f(x - h) + k$



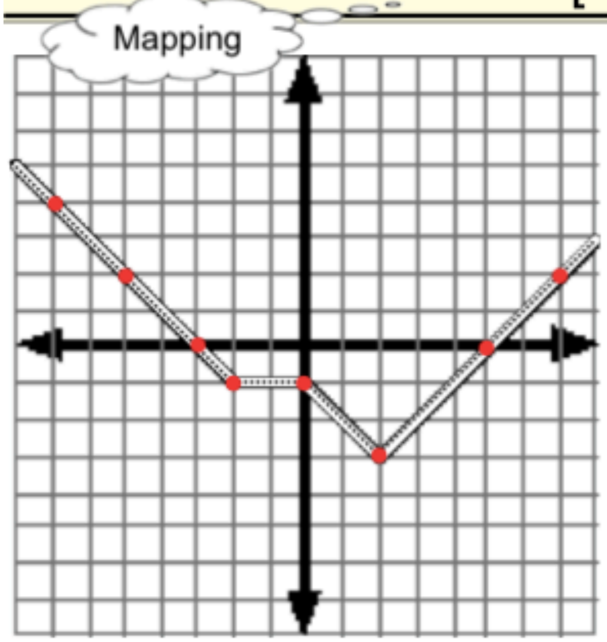
TRY: If the point (2, –3) is a point on the graph of the function y = f(x), determine the new coordinates of the translated image of this point on the function y = f(x + 5) + 1

★ Mapping

TRY: Determine the translation that has been applied to the graph f(x) to obtain the graph of g(x). Determine the equation of the translated function in the form $y - k = f(x - h)$.



TRY: Given the graph of the function y = f(x) below, sketch the graph of y = f(x – 3) – 2 [**you may see it written like: y + 2 = f(x – 3)**]



[*hint:* touch]

Given *f*(*x*) [*base points*]		**New *f*(*x*)**	
x	*y*		
-7	4		
-5	2		
-3	0		
-2	-1		
0	-1		
2	-3		
5	0		
7	2		

Complete Assignment Questions #1 - #9

Assignment

1. Describe how the graphs of the following functions relate to the graph of $y = f(x)$.

   a) $y = f(x + 9)$ **b)** $y = f(x) + 7$ **c)** $y = f(x - 4) + 4$

   d) $y - 6 = f(x)$ **e)** $y = 3 + f(x - 5)$ **f)** $y + 2 = f(x + 3) - 10$

2. Write the equation of the image of $y = f(x)$ after each transformation.

   a) A vertical translation of 10 units down.

   b) A horizontal translation of 8 units right and a vertical translation of 9 units up.

   c) A translation of t units up and s units left.

3. The function $y = f(x)$ is transformed to $y = f(x - h) + k$. Find the values of h and k for the following translations.

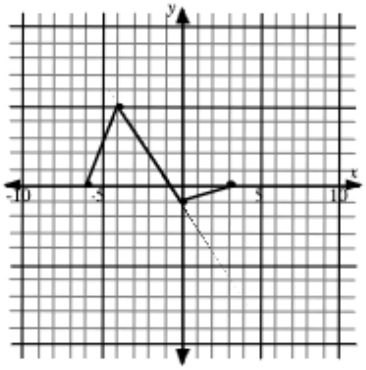
   a) 7 units right **b)** 4 units up and 2 units left **c)** a units right and b units down.

4. The point (–3, 5) lies on the graph of $y = f(x)$. State the coordinates of the image of this point under the following transformations.

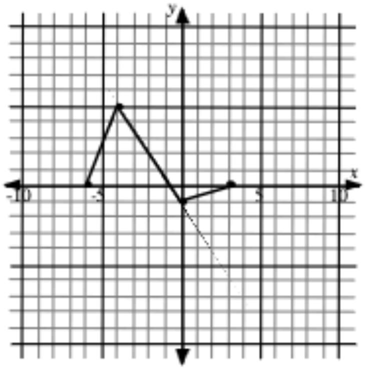
   a) $y = f(x) + 3$ **b)** $y + 5 = f(x + 2)$ **c)** $(x, y) \rightarrow (x - 7, y - 1)$

5. Given the graph of the function $y = f(x)$ sketch the graph of the indicated function.

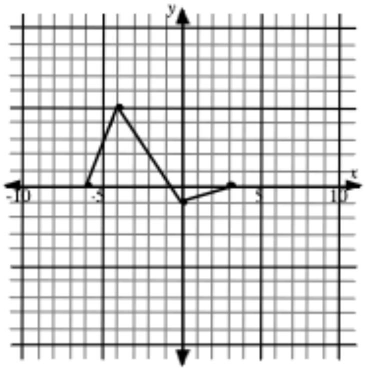
a) $y = f(x - 4)$



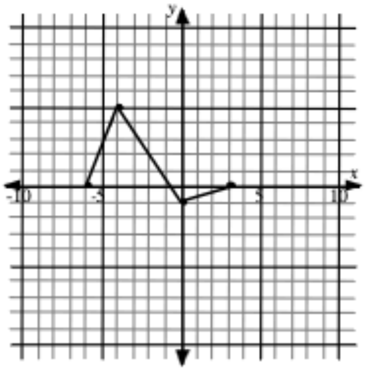
b) $y - 3 = f(x)$



c) $y = f(x + 2) - 3$



d) $y + 2 = f(x - 5)$



6. What happens to the graph of the function $y = f(x)$ if you make these changes to its equation?

a) replace x with $x - 8$

b) replace y with $y + 2$

c) replace x with $x + 4$ and y with $y - 7$

7. The function $y = f(x)$ is transformed to $y = f(x + 2) + 4$. If the point $(3, -1)$ lies on the graph of $y = f(x)$, which of the following points must lie on the graph of $y = f(x + 2) + 4$?

A. $(5, 3)$
B. $(1, 3)$
C. $(7, 1)$
D. $(7, -3)$

8. The function $y = f(x)$ is transformed to $y - 3 = f(x - 1)$. If the point $(-2, 4)$ lies on the graph of $y - 3 = f(x - 1)$, which of the following points must lie on the graph of $y = f(x)$?

A. $(-1, 7)$
B. $(-1, 1)$
C. $(-3, 7)$
D. $(-3, 1)$

9. The graph of $y = g(x)$ was transformed to the graph of $y = g(x - 7) + 2$. Which of the following statements describes the transformation?

A. The graph of $y = g(x)$ has been translated 2 units to the right and 7 units upward.
B. The graph of $y = g(x)$ has been translated 7 units to the left and 2 units downward.
C. The point (x, y) on the graph $y = g(x)$ has been translated to point $(x + 7, y + 2)$.
D. The point (x, y) on the graph $y = g(x)$ has been translated to point $(x - 7, y - 2)$.

Answer Key

1. **a)** horizontal translation 9 units left **b)** vertical translation 7 units up
c) translation 4 units right and 4 units up **d)** vertical translation 6 units up
e) translation 5 units right and 3 units up **f)** translation 3 units left and 12 units down

2. **a)** $y = f(x) - 10$ **b)** $y = f(x - 8) + 9$ **c)** $y = f(x + s) + t$

3. **a)** $h = 7, k = 0$ **b)** $h = -2, k = 4$ **c)** $h = a, k = -b$

4. **a)** $(-3, 8)$ **b)** $(-5, 0)$ **c)** $(-10, 4)$

5. **a)** the graph is translated 4 units right **b)** the graph is translated 3 units up
c) the graph is translated 2 units left and 3 units down
d) the graph is translated 5 units right and 2 units down

6. **a)** horizontal translation 8 units right **b)** vertical translation 2 units down
c) translation 4 units left and 7 units up

7. B **8.** D **9.** C

Complete Assignment Questions #1 - #8

Assignment

1. Describe how the graph of the second function compares to the graph of the first function.

a) $y = x^3$
$y = x^3 - 1$

b) $y = 7x - 1$
$y = 7(x - 3) - 1$

c) $y = \cos x°$
$y = \cos (x + 45)°$

d) $y = |x|$
$y + 3 = |x + 6|$

e) $y = \dfrac{1}{x^2 + 1}$
$y - 2 = \dfrac{1}{(x - 3)^2 + 1}$

f) $y = a^x$
$y = a^{x+1} + 1$

2. Write the equation of the image of:

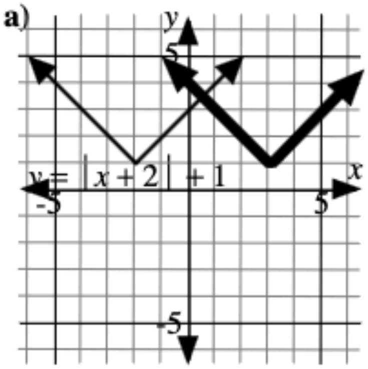
a) $y = x^4$ after a horizontal translation of 2 units to the left.

b) $y = 2|x|$ after a translation of 3 units down and 1 unit left.

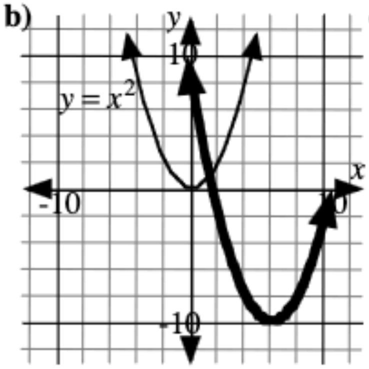
c) $y = \dfrac{1}{\sqrt{x}}$ after a horizontal translation of 3 units to the right and a vertical translation of 2 units up.

3. The function represented by the thick line is a transformation of the function represented by the thin line. Write an equation for each function represented by the thick line.

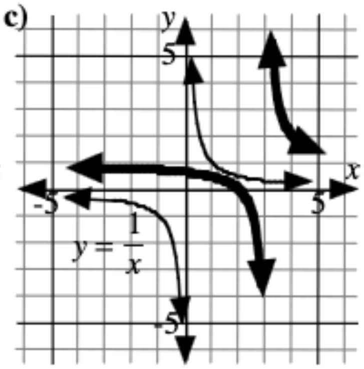
a)



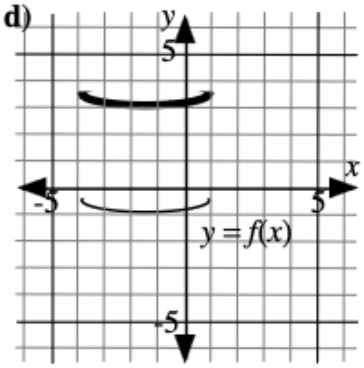
b)



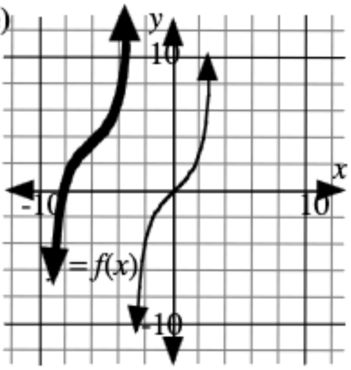
c)



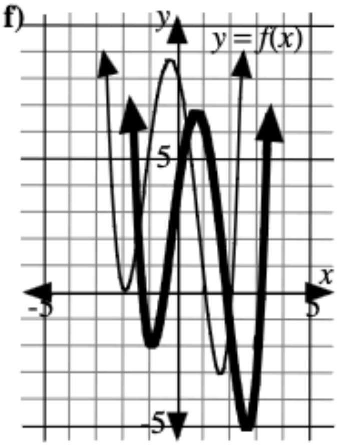
d)



e)



f)



4. a) What vertical translation would be applied to $y = x^2$ so that the translation image passes through $(3, 5)$?

b) What horizontal translation would be applied to $y = x^3 + 1$ so that the translation image passes through $(5, 28)$?

c) What horizontal translation would be applied to $y = \dfrac{1}{x-3}$ so that the translation image passes through $\left(1, \dfrac{1}{2}\right)$?

5. On a certain route into town, shuttle buses depart every 15 minutes from 06:30 until 07:30. The distance d, in kilometres, they travel can be described as a function of time, t, in hours, and represented by the equation $d = f(t) = 60t$.

If $t = 0$ at 06:30, write an equation which represents the distance travelled by:

a) the second bus **b)** the third bus **c)** the last bus

6. The graph of the function $y = f(x)$ passes through the point $(4, 7)$. Under a transformation, the point $(4, 7)$ is transformed to $(6, 6)$. A possible equation for the transformed function is

A. $y - 1 = f(x + 2)$
B. $y - 2 = f(x + 1)$
C. $y + 1 = f(x - 2)$
D. $y + 2 = f(x - 1)$

7. The function $f(x) = \sqrt{x} + 5$ is transformed by a translation of 2 units down and 4 units to the left. The transformed function passes through the point $(20, y)$.
To the nearest tenth, the value of y is _____ .

(Record your answer in the numerical response box from left to right)

8. The function $r(x) = \dfrac{1}{x+3}$ is transformed by a translation of 3 units up and 5 units to the right. The transformed function passes through the point $(x, 7)$.
The value of x to the nearest hundredth is _____ .

(Record your answer in the numerical response box from left to right)

Answer Key

1. a) vertical translation 1 unit down **b)** translation 3 units right
c) horizontal translation 45° left **d)** translation 6 units left and 3 units down
e) translation 3 units right and 2 units up **f)** translation 1 unit left and 1 unit up

2. a) $y = (x+2)^4$ **b)** $y = 2|x+1| - 3$ **c)** $y = \dfrac{1}{\sqrt{x-3}} + 2$

3. a) $y = |x-3| + 1$ **b)** $y = (x-6)^2 - 10$ **c)** $y = \dfrac{1}{x-3} + 1$
d) $y = f(x) + 4$ **e)** $y = f(x+6) + 4$ **f)** $y = f(x-1) - 2$

4. a) vertical translation 4 units down **b)** horizontal translation 2 units right
c) horizontal translation 4 units left

5. a) $d = 60\left(t - \dfrac{1}{4}\right)$ **b)** $d = 60\left(t - \dfrac{1}{2}\right)$ **c)** $d = 60(t-1)$

6. C

7.

7	.	9	

8.

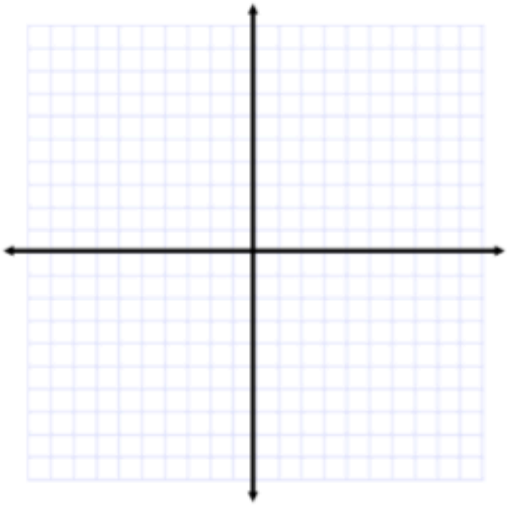
2	.	2	5

Topic 2 Reflections & Stretches

EXAMPLE 1 - **Comparing** y = f(x) **to** y = -f(x)

Clear all functions. Enter $y_1 = \sqrt{x}$ $y_2 = -\sqrt{x}$

Sketch graphs.

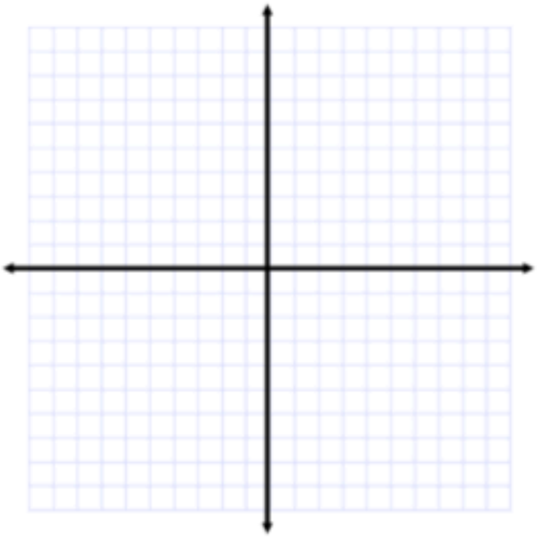


What happens to the function?

EXAMPLE 1b - **Comparing** y = f(x) **to** y = f(-x)

Clear all functions. Enter $y_1 = \sqrt{x}$ $y_2 = \sqrt{-x}$

Sketch graphs.



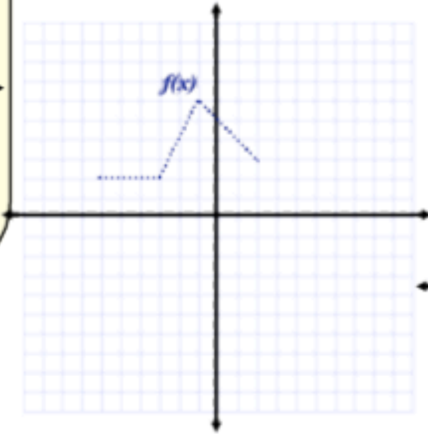
What happens to the function?

TRY: Given the graph of y = f(x), graph the function y = -f(x) and y = f(-x).

y = f(x)

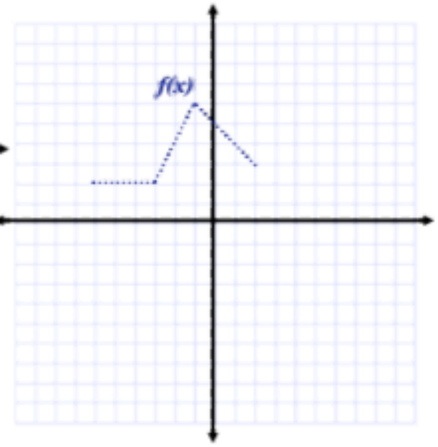
y = -f(x)

Reflection on ____ axis



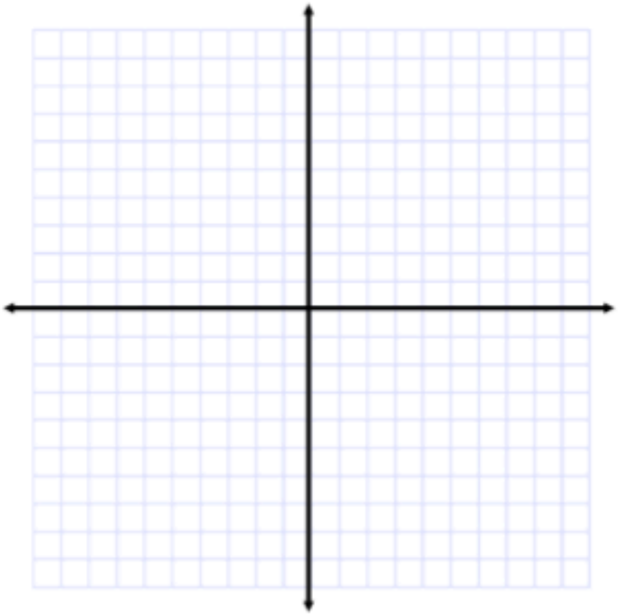
y = f(-x)

Reflection on ____ axis



TRY: a) Graph the function $y = \sqrt{x} + 2$

b) Using this function sketch the graphs y = f(-x) and y = -f(x)



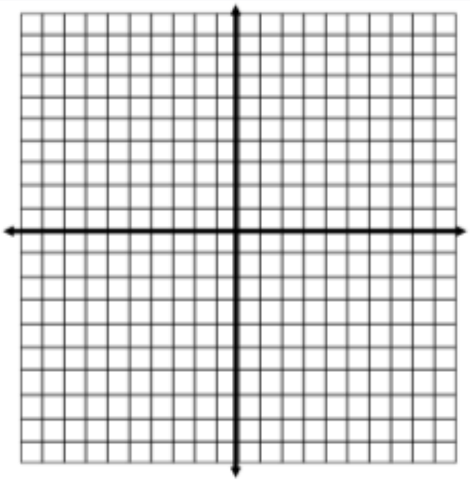
EXAMPLE 1c - **Comparing** $y = f(x)$ **to** $x = f(y)$

Clear all functions.

1. Enter $y_1 = 2x + 1$
   $x = 2y + 1$ (Solve for y)

2. Enter $y_2 =$ .....

3. Enter $y_3 = x$

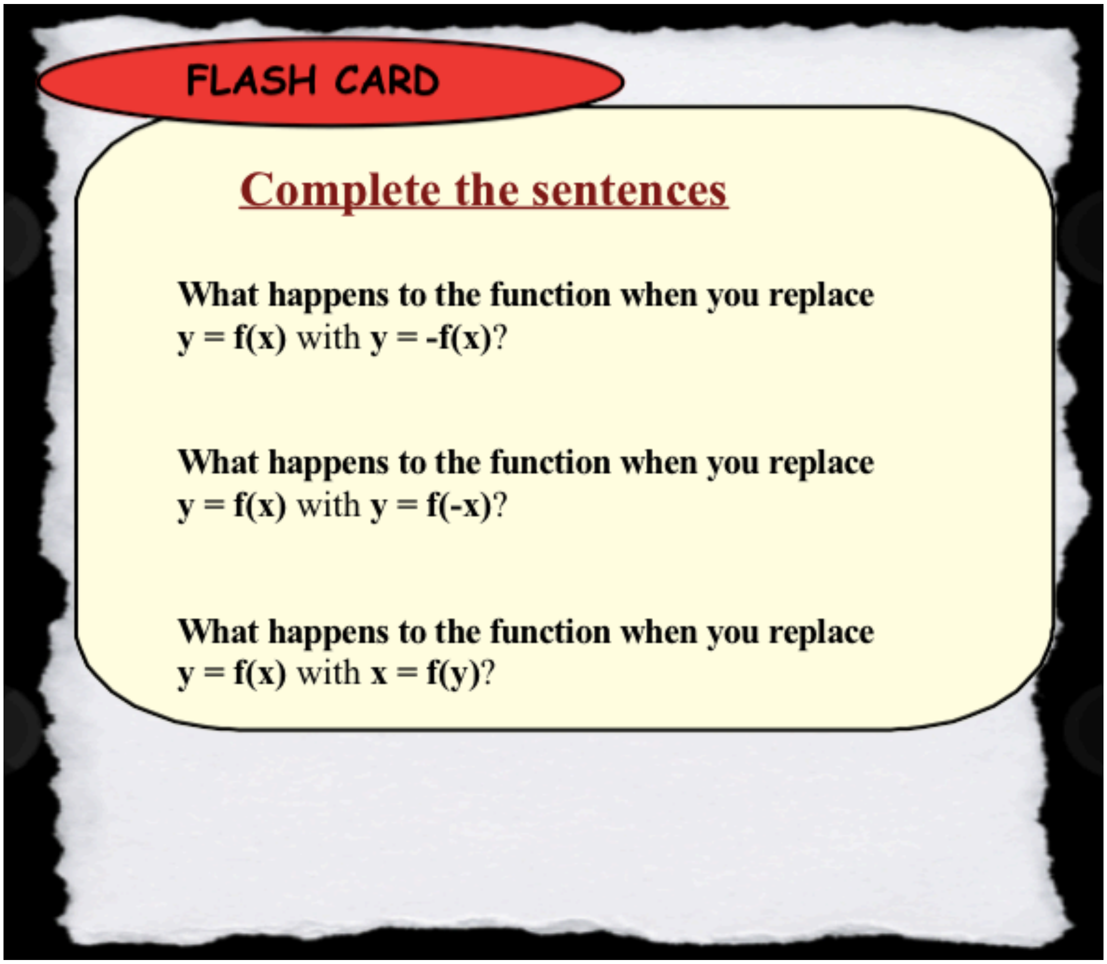


What happens to $y = f(x)$ when *x* and *y* are interchanged?

These functions are *inverses* of each other.
Note: *$y = f^{-1}(x)$ is sometimes used to represent the equation of the inverse.*

TRY: Let $y = f(x)$, find the equation of the $x = f(y)$ for:

a) $f(x) = (x + 5)^2 + 1$ b) $f(x) = \sqrt{(x - 3)} + 4$



FLASH CARD

Complete the sentences

What happens to the function when you replace $y = f(x)$ with $y = -f(x)$?

What happens to the function when you replace $y = f(x)$ with $y = f(-x)$?

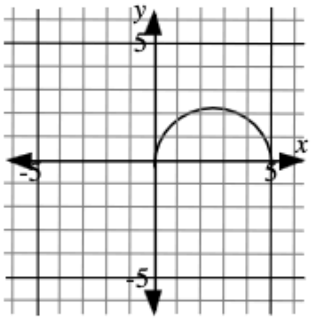
What happens to the function when you replace $y = f(x)$ with $x = f(y)$?

Complete Assignment Questions #1 - #7

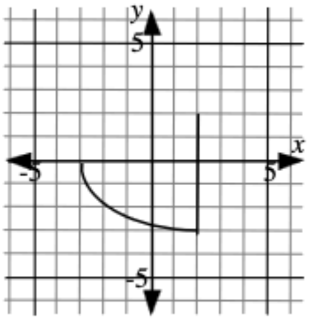
Assignment

1. The graph of $y = f(x)$ is shown. Sketch the graph of $y = -f(x)$.

a)

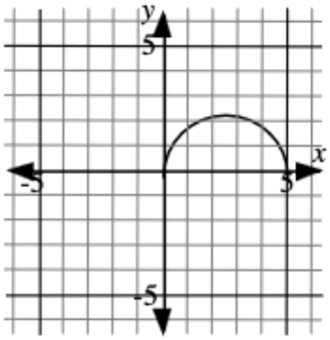


b)

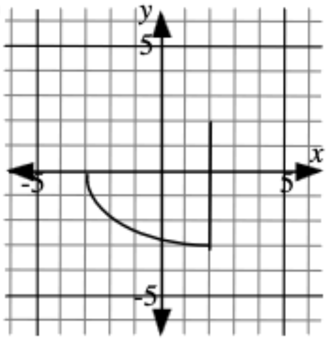


2. The graph of $y = f(x)$ is shown. Sketch the graph of $y = f(-x)$.

a)

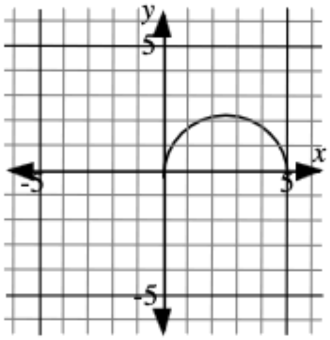


b)

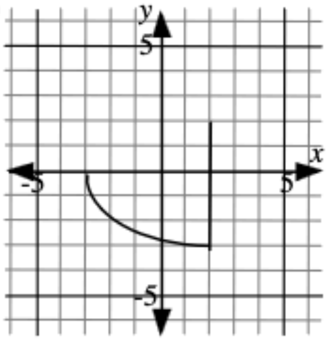


3. The graph of $y = f(x)$ is shown. Sketch the graph of $x = f(y)$.

a)

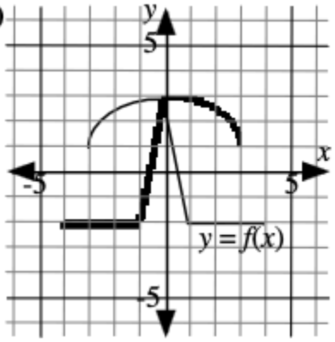


b)

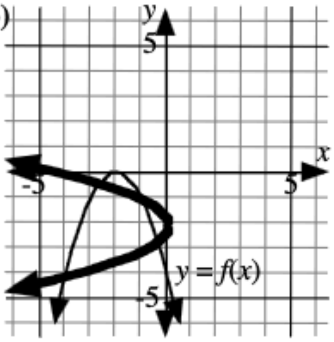


4. The graph drawn in the thick line is a transformation of the graph drawn in the thin line. Write an equation for each graph drawn in the thick line and state whether this graph represents a function.

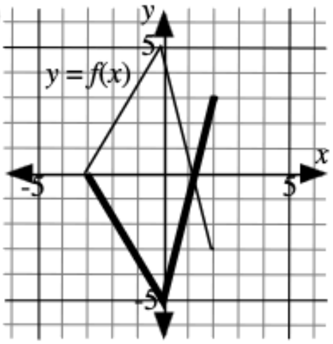
a)



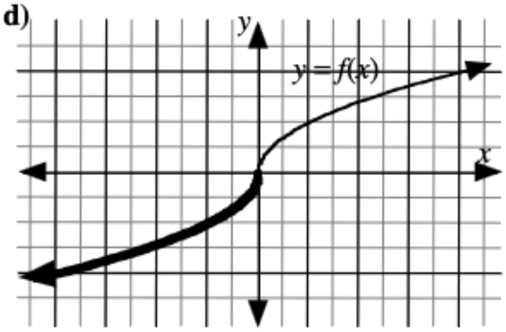
b)



c)



d)



5. The function $y = f(x)$ is transformed to the function below. Given that there are invariant points, describe the location of these points.

a) $y = -f(x)$ **b)** $y = f(-x)$ **c)** $x = f(y)$

6. The point (x, y) lies of the graph of the function $y = f(x)$. State the coordinates of the image of (x, y) under the following transformations:

a) $y = -f(x)$ **b)** $y = f(-x)$ **c)** $x = f(y)$

7. Consider the graph of the function $f(x) = x^2$. Which of the following transformations would result in an identical graph?

A. $-f(x)$

B. $f(-x)$

C. $-f(-x)$

D. $f(x + 1)$

Answer Key

1. **a)** and **b)** graph is reflected in x-axis **2.** **a)** and b) graph is reflected in y-axis
3. **a)** and **b)** graph is reflected in the line $y = x$
4. **a)** $y = f(-x)$ is a function **b)** $x = f(y)$ is a not a function
c) $y = -f(x)$ is a function **d)** $y = -f(-x)$ is a function
5. **a)** on the x-axis **b)** on the y-axis **c)** on the line $y = x$
6. **a)** $(x, -y)$ **b)** $(-x, y)$ **c)** (y, x) **7.** B

Complete Assignment Questions #1 - #4

Assignment

1. Write the equation of the image of:

a) $y = \frac{1}{x}$ after a reflection in the line $y = x$

b) $y = x^3 + x$ after a reflection in the y-axis

c) $y = |x|$ after a reflection in the x-axis.

d) $y = \sqrt{x - 2}$ after a reflection in the line $y = x$

e) $y = x^2 + 1$ after a reflection in the y-axis

f) $y = \cos x$ after a reflection in the x-axis

2. Describe how the graph of the second function compares to the graph of the first function.

a) $y = 3x + 1$
$y = -3x - 1$

b) $y = 3x + 1$
$y = -3x + 1$

c) $y = 3x + 1$
$x = 3y + 1$

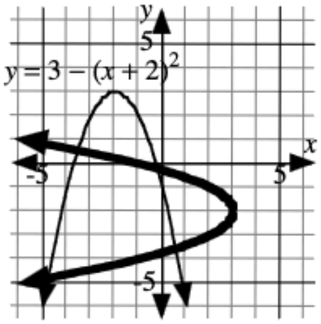
d) $y = 10^x$
$y = 10^{-x}$

e) $y = 10^x$
$y = -10^x$

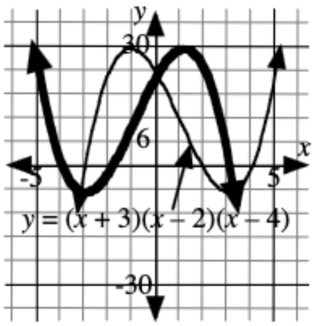
f) $y = 4x^2$
$y = \pm\frac{\sqrt{x}}{2}$

3. The graph drawn in the thick line is a transformation of the graph drawn in the thin line. Write an equation for each graph drawn in the thick line.

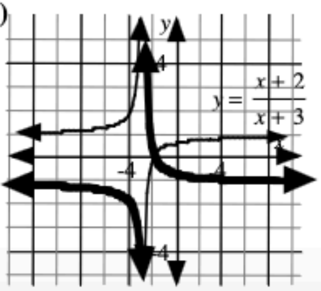
a)



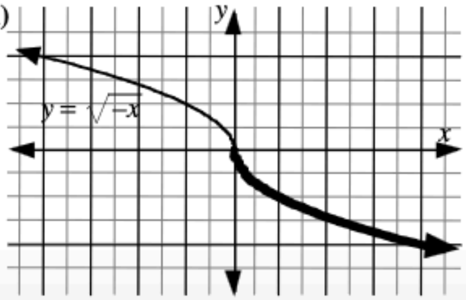
b)



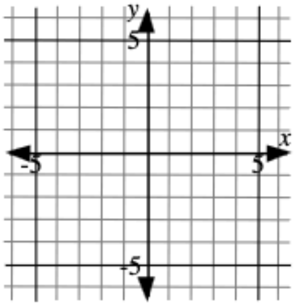
c)



d)



4.a) Sketch the graph of $f(x) = (x - 1)^2$.



b) Write the equation for:

i) $y = -f(x)$

ii) $y = f(-x)$

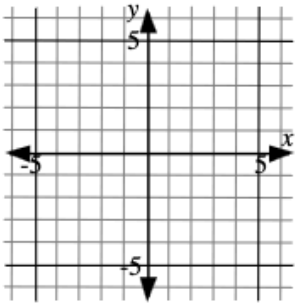
iii) $x = f(y)$

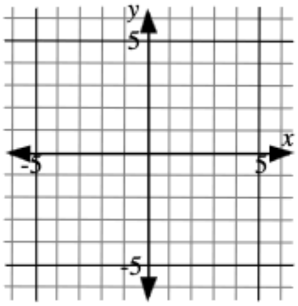
c) Sketch each graph in b) and state whether the graph represents a function.

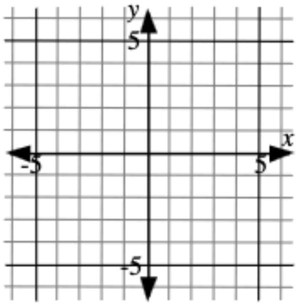
i) $y = -f(x)$

ii) $y = f(-x)$

iii) $x = f(y)$







Answer Key

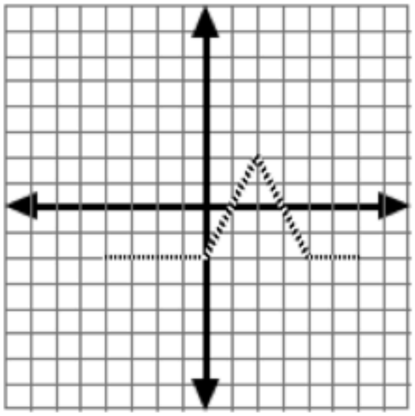
1.a) $x = \frac{1}{y}$ or $y = \frac{1}{x}$ **b)** $y = -x^3 - x$ **c)** $y = -|x|$
d) $x = \sqrt{y-2}$ or $y = x^2 + 2, x \geq 0$ **e)** $y = x^2 + 1$ **f)** $y = -\cos x$

2.a) reflection in the x-axis **b)** reflection in the y-axis **c)** reflection in the line $y = x$
d) reflection in the y-axis **e)** reflection in the x-axis **f)** reflection in the line $y = x$
3.a) $x = 3 - (y+2)^2$ or $y = \pm\sqrt{3-x} - 2$ **b)** $y = (-x+3)(-x-2)(-x-4)$ or $y = -(x-3)(x+2)(x+4)$
c) $y = -\frac{x+2}{x+3}$ **d)** $y = -\sqrt{x}$
4.a) parabola opening up with vertex (1, 0)
b) **i)** $y = -(x-1)^2$ **ii)** $y = (-x-1)^2$ or $y = (x+1)^2$ **iii)** $x = (y-1)^2$
c) **i)** parabola opening down with vertex (1, 0). Is a function. **ii)** parabola opening up with vertex (–1, 0). Is a function. **iii)** parabola opening right with vertex (0, 1). Is not a function.

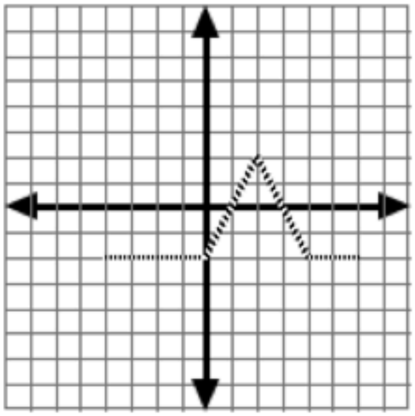
LG 1 Worksheet A (Sketching Transformations)

1. Given the dotted $y = f(x)$ below:

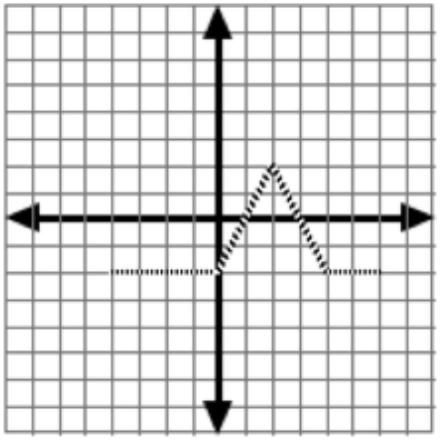
a) Find $y = f(x+3) + 5$



b) Find $\frac{y}{3} = f(2x)$

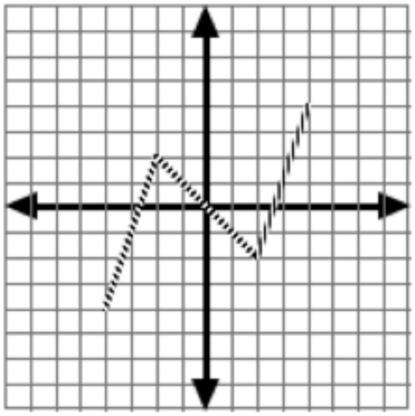


c) Find $y = -f(-x)$

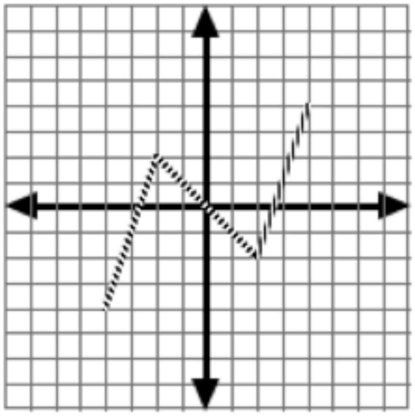


2. Given the dotted $y = f(x)$ below:

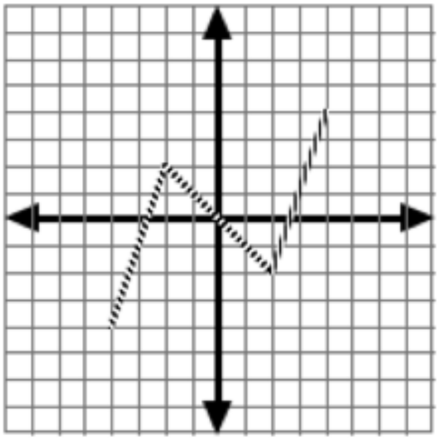
a) Find $y + 3 = f(x-2)$



b) Find $y = 2f(2x)$

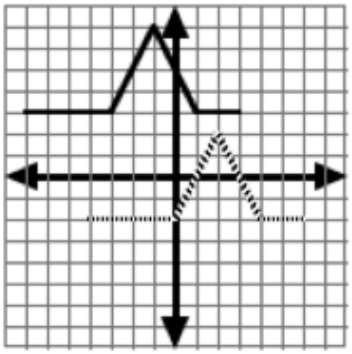


c) Find $-2y = f\left(\frac{1}{2}x\right)$

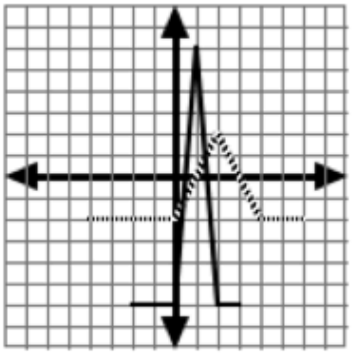


Answer Key

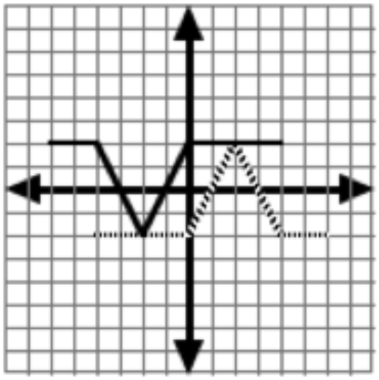
1. a.



b.

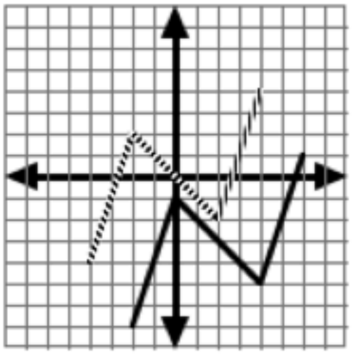


c.

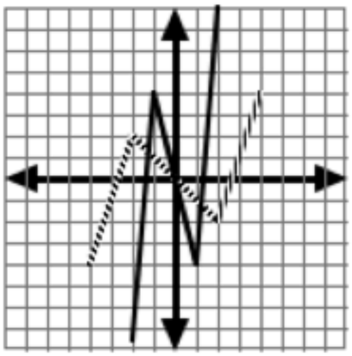


Answer Key

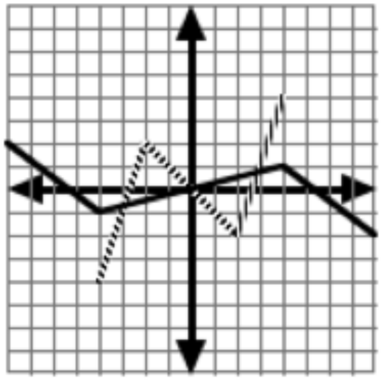
2. a.



b.



c.



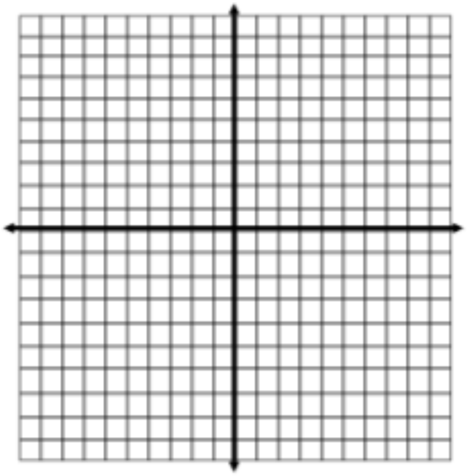
EXAMPLE 2. Comparing y = f(x) to y = af(x)

Enter $y_1 = |x|$ $y_2 = \frac{1}{2}|x|$

Vertical compression

☆ *Vertical stretch* by a factor of $\frac{1}{2}$

Sketch graphs



To sketch new graph take base points from $y = |x|$

ex. ⇓ , ⇒ Now take $\frac{1}{2}$ of the y'

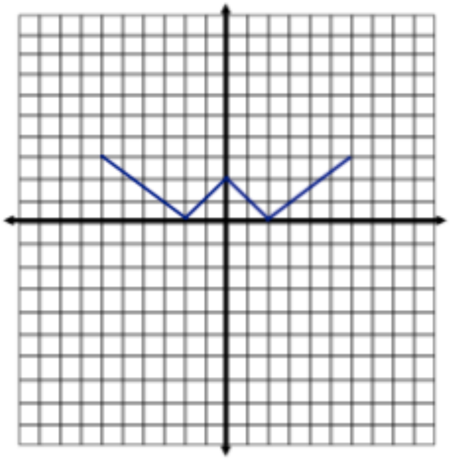
x	*y*		
0	0		
2	2		
-2	2		
4	4		
-4	4		

TRY: Given the graph of y = f(x), sketch g(x) = 2f(x).

- Describe the transformation: _____________ *stretch* by a factor of ____ .
- State the domain: __________
- Range : ____________

Vertical expansion

Sketch graphs



To sketch new graph take base points from y = f(x)

ex. ⇓ , ⇒ Now times the y's

x	*y*		
-6	3		
-2	0		
0	2		
2	0		
6	3		

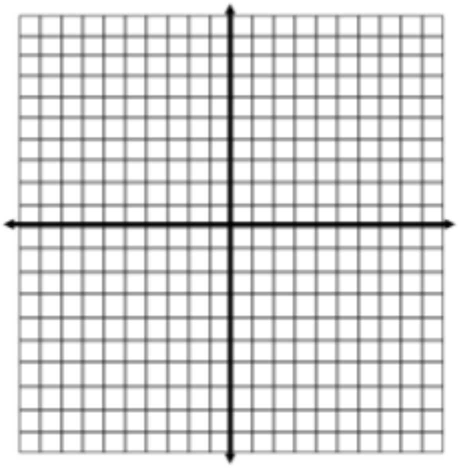
EXAMPLE 3. Comparing $y = f(x)$ to $y = f(bx)$

Enter $y_1 = x^2$ $y_2 = (2x)^2$

Horizontal compression

☆ *Horizontal stretch* by a factor of $\frac{1}{2}$

Sketch graphs



To sketch new graph take base points from $y = x^2$

ex. ⇓ , ⇒ Now take $\frac{1}{2}$ of the x's

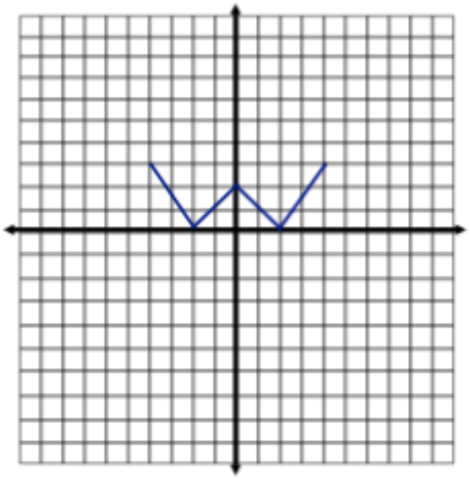
x	y		
0	0		
1	1		
-1	1		
2	4		
-2	4		

TRY: Given the graph of $y = f(x)$, sketch $g(x) = f(\frac{1}{2}x)$.

- Describe the transformation: _____________ *stretch* by a factor of ____ .
- State the domain: ___________
- Range : _____________

Horizontal expansion

Sketch graphs



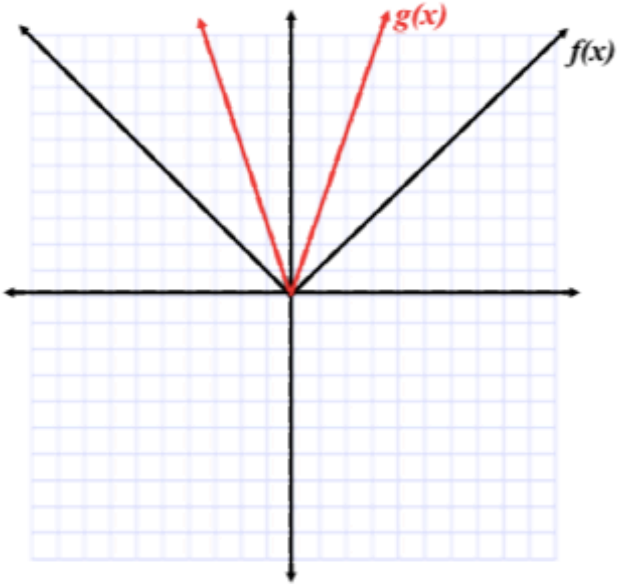
To sketch new graph take base points from $y = f(x)$

ex. ⇓ , ⇒ Now times the x's

x	y		
-4	3		
-2	0		
0	2		
2	0		
4	3		

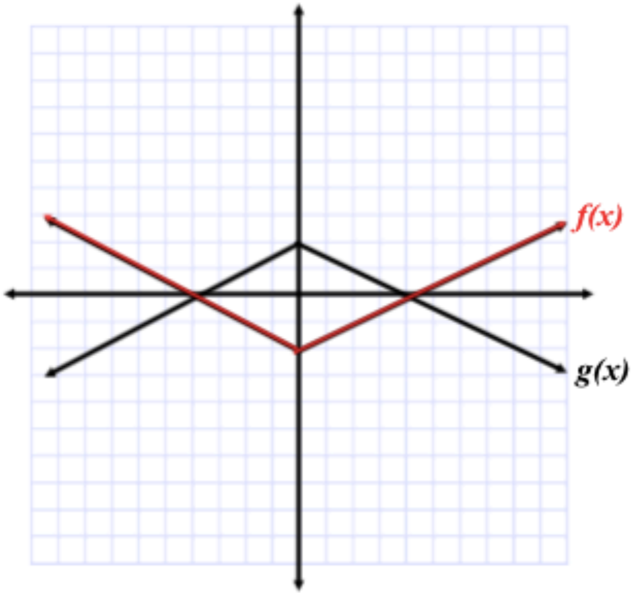
EXAMPLE 4. Writing the Equation of a Transformed Function

The graph of the function f(x) has been transformed by either a stretch or a reflection. Write the equation of the transformed graph, g(x).



☛ start with:
$g(x) = ag(bx - h) \pm k$

TRY: The graph of the function f(x) has been transformed by either a stretch or a reflection. Write the equation of the transformed graph, g(x).

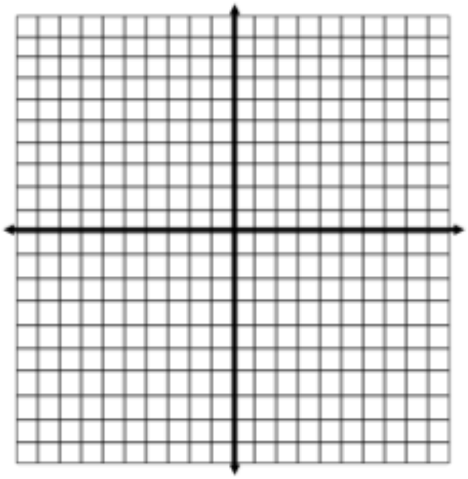


☛ start with:
$g(x) = ag(bx - h) \pm k$

TRY: Sketch $y = \sqrt{x}$ and the following functions on the same grid.

a) $y = 2\sqrt{x}$ b) $y = \sqrt{\frac{1}{3}(x)}$ c) $y = 2\sqrt{2x}$

hint: base points



What occurred in each?

TRY: Describe what happens to the function $y = x^2$ when you make each change to equation and give the coordinates of the image of the point (2, 4) under each change:

a) Replace $y = x^2$ with $y = (2x)^2$

b) Replace $y = x^2$ with $y = 3x^2$

TRY: Let $y = f(x)$, find the equation of the $x = f(y)$ for:

a) $f(x) = (x + 5)^2 + 1$ b) $f(x) = \sqrt{(x - 3)} + 4$

TRY: For $y = x^3$, write an equation using the following descriptions.

a) Expanded horizontally by a factor of 2

b) Compressed vertically by a factor of $\frac{1}{3}$

TRY: Given the graph of the function y = f(x), sketch the following functions:

Mapping

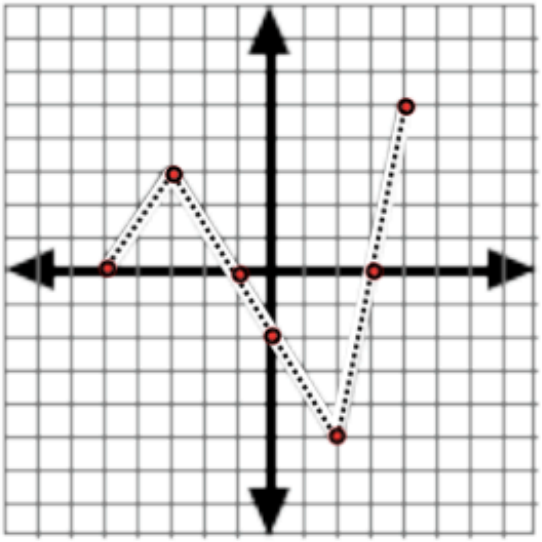
a) $y = f(2x)$

hint: base points

[touch]

b) $y = \frac{1}{3}f(x)$

bp		(a)		(b)	
x	*y*				
-5	0				
-3	3				
-1	0				
0	-2				
2	-5				
3	0				
4	5				



Assignment

Complete Assignment Questions #1 - #7

1. Write the replacement for x or y and write the equation of the image of $y = f(x)$ after each transformation.

a) A horizontal stretch by a factor of 3 about the y-axis.

b) A vertical stretch by a factor of 6 about the x-axis.

c) A horizontal stretch about the y-axis by a factor of $\frac{5}{7}$.

d) A vertical stretch about the x-axis by a factor of $\frac{2}{3}$.

e) A reflection in the y-axis and a horizontal stretch by a factor of 3 about the y-axis.

f) A reflection in the x-axis and a vertical stretch by a factor of $\frac{3}{4}$ about the x-axis.

g) A reflection in the y-axis and a horizontal stretch about the y-axis by a factor of $\frac{3}{4}$.

h) A horizontal stretch about the y-axis by a factor of 4 and a vertical stretch about the x-axis by a factor of 4.

i) A horizontal stretch about the y-axis by a factor of 0.5, a vertical stretch by a factor of 2 about the x-axis and a reflection in the x-axis.

2. The function $y = f(x)$ is transformed to $y = af(bx)$. Determine the values of a and b for:

a) A horizontal stretch by a factor of $\frac{2}{3}$ about the y-axis.

b) A vertical stretch about the x-axis by a factor of 5.

c) A horizontal stretch about the y-axis by a factor of $\frac{5}{2}$ and a reflection in the y-axis.

d) A vertical stretch about the x-axis by a factor of $\frac{1}{3}$, a horizontal stretch about the y-axis by a factor of $\frac{1}{10}$ and a reflection in the y-axis.

3. Consider the function $f(x) = x^2$.

a) Determine the equation of the image of the function if it is stretched vertically by a factor of 4 about the x-axis.

b) Determine the equation of the image of the function if it is stretched horizontally by a factor of $\frac{1}{2}$ about the y-axis.

c) What do you notice?

d) Give an example of a function where the stretches in a) and b) would not result in the same image.

4. **a)** What information about the graph of $y = f(kx)$ does k provide?

b) What information about the graph of $ky = f(x)$ does k provide?

c) What information about the graph of $y - k = f(x)$ does k provide?

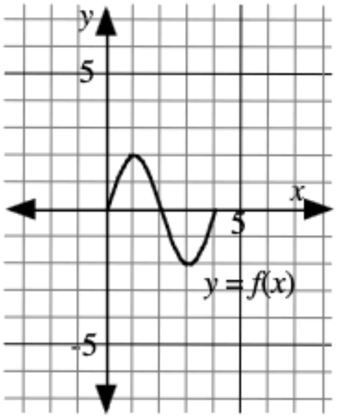
d) What information about the graph of $y = f(x - k)$ does k provide?

e) What information about the graph of $y = kf(x)$ does k provide?

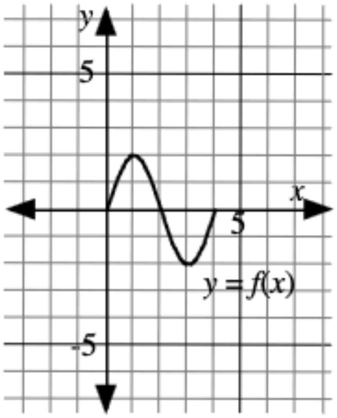
5. The graph of $y = f(x)$ is shown. In each case:

- **i)** sketch the graph of the transformed function
- **ii)** state the domain and range of the transformed function
- **iii)** state the coordinates of any invariant points.

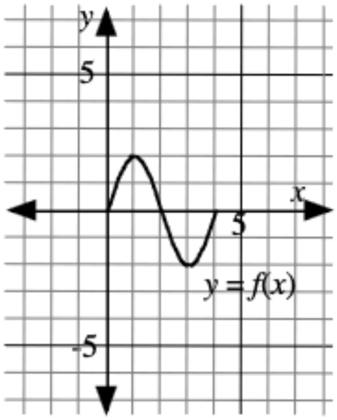
a) $y = f(2x)$



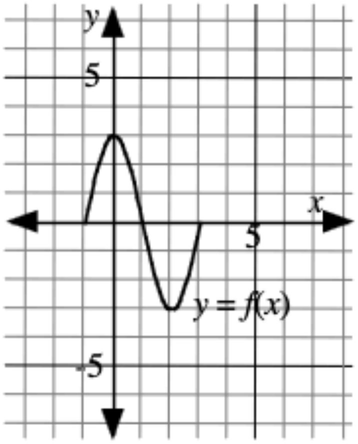
b) $y = -2f(x)$



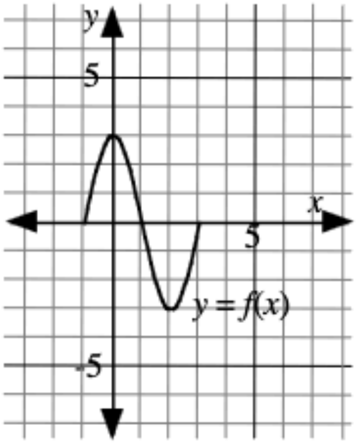
c) $y = \frac{1}{2}f\left(\frac{1}{2}x\right)$



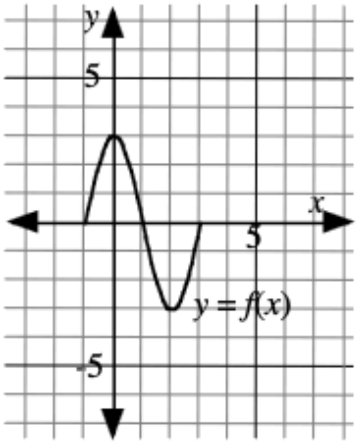
d) $y = f(2x)$



e) $y = -2f(x)$



f) $y = \frac{1}{2}f\left(\frac{1}{2}x\right)$



6. What happens to the graph of the function $y = f(x)$ if the following replacements are made?

a) Replace x with $\frac{1}{2}x$.

b) Replace y with $4y$.

c) Replace y with $-2y$ and x with $4x$.

d) Replace y with $y - 4$ and x with $-\frac{1}{4}x$.

7. The graph of $y = f(x)$ is stretched vertically by a factor of $\frac{1}{2}$ about the x-axis, stretched horizontally by a factor of $\frac{1}{4}$ about the y-axis, and reflected in the y-axis. If the equation of the image is written in the form $y = af(bx)$, the value of $a - b$, to the nearest tenth, is _____ .

(Record your answer in the numerical response box from left to right) ☐☐☐☐

Answer Key

1. **a)** $x \rightarrow \frac{1}{3}x,\ y = f\left(\frac{1}{3}x\right)$ **b)** $y \rightarrow \frac{1}{6}y,\ y = 6f(x)$ **c)** $x \rightarrow \frac{7}{5}x,\ y = f\left(\frac{7}{5}x\right)$

d) $y \rightarrow \frac{3}{2}y,\quad y = \frac{2}{3}f(x)$ **e)** $x \rightarrow -\frac{1}{3}x,\ y = f\left(-\frac{1}{3}x\right)$ **f)** $y \rightarrow -\frac{4}{3}y,\ y = -\frac{3}{4}f(x)$

g) $x \rightarrow -\frac{4}{3}x,\ y = f\left(-\frac{4}{3}x\right)$ **h)** $x \rightarrow \frac{1}{4}x$ and $y \rightarrow \frac{1}{4}y,\ y = 4f\left(\frac{1}{4}x\right)$

i) $x \rightarrow 2x$ and $y \rightarrow -\frac{1}{2}y,\quad y = -2f(2x)$

2. **a)** $a = 1 \quad b = \frac{3}{2}$ **b)** $a = 5 \quad b = 1$ **c)** $a = 1 \quad b = -\frac{2}{5}$ **d)** $a = \frac{1}{3} \quad b = -10$

3. **a)** $y = 4f(x) = 4x^2$ **b)** $y = (2x)^2 = 4x^2$ **c)** both transformations result in the same image
d) many possible answers including $f(x) = x, f(x) = x^3, f(x) = x^2 + 1$, etc.

4. **a)** horizontal stretch about the y-axis by a factor of $\frac{1}{k}$

b) vertical stretch about the x-axis by a factor of $\frac{1}{k}$

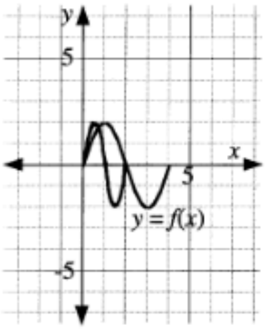
c) vertical translation of k units: up if $k > 0$, down if $k < 0$

d) horizontal translation of k units: right if $k > 0$, left if $k < 0$

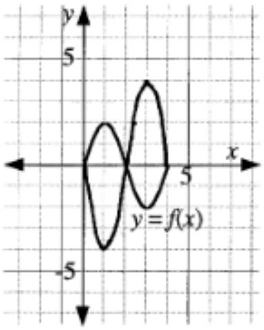
e) vertical stretch about the x-axis by a factor of k

5.

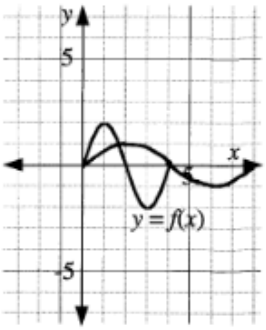
a)



b)

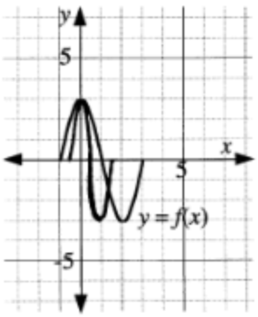


c)

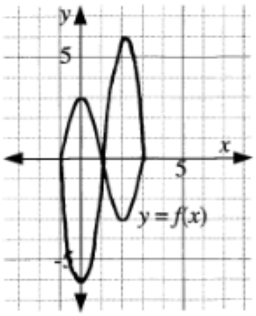


	a)	b)	c)
Domain:	$\{x \mid 0 \le x \le 2, x \in \Re\}$	$\{x \mid 0 \le x \le 4, x \in \Re\}$	$\{x \mid 0 \le x \le 8, x \in \Re\}$
Range:	$\{y \mid -2 \le y \le 2, y \in \Re\}$	$\{y \mid -4 \le y \le 4, y \in \Re\}$	$\{y \mid -1 \le y \le 1, y \in \Re\}$
Invariant Points:	$(0, 0)$	$(0, 0)$, $(2, 0)$, $(4, 0)$	$(0, 0)$

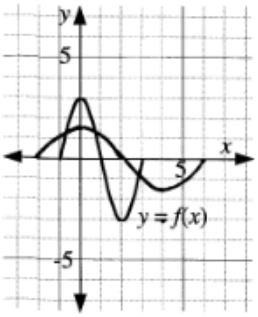
d)



e)



f)



	d)	e)	f)
Domain:	$\left\{x \mid -\frac{1}{2} \le x \le \frac{3}{2}, x \in \Re\right\}$	$\{x \mid -1 \le x \le 3, x \in \Re\}$	$\{x \mid -2 \le x \le 6, x \in \Re\}$
Range:	$\{y \mid -3 \le y \le 3, y \in \Re\}$	$\{y \mid -6 \le y \le 6, y \in \Re\}$	$\left\{y \mid -\frac{3}{2} \le y \le \frac{3}{2}, y \in \Re\right\}$
Invariant Points:	$(0, 3)$	$(-1, 0)$, $(1, 0)$, $(3, 0)$	none

6. **a)** horizontal stretch about the y-axis by a factor of 2

b) vertical stretch about the x-axis by a factor of $\frac{1}{4}$

c) horizontal stretch about the y-axis by a factor of $\frac{1}{4}$, vertical stretch about the x-axis by a factor of $\frac{1}{2}$, and a reflection in the x-axis.

d) horizontal stretch about the y-axis by a factor of 4, a reflection in the y-axis, followed by a vertical translation of 4 units up.

7.

4	.	5	

LG 1 QUIZ (Transformations)

1. Describe in words how the graph of the following function can be found from the graph of $y = f(x)$:

$y - 5 = -f(x + 4)$

---

2. If (-2, 5) is a point on the graph of $y = g(x)$, find a point on the graph of $y = g(x + 3) - 4$.

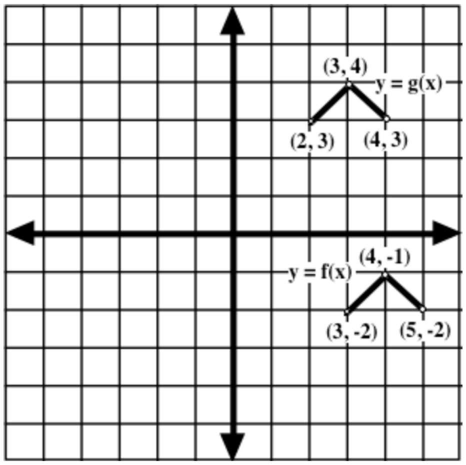
---

3. The graph of $y = x^3$ is translated 5 units to the left and 4 units down to form the transformed function $y = g(x)$. Determine the equation of the function $y = g(x)$.

4. **The domain of the function $y = h(x)$ is $-2 < x \leq 6$ and the range is $4 < y \leq 10$. Find the domain and range of the function $y - 1 = -h(x + 6)$.**

---

5. **Given the graph of $y = f(x)$ below, find the equation of the transformed function $y = g(x)$.**



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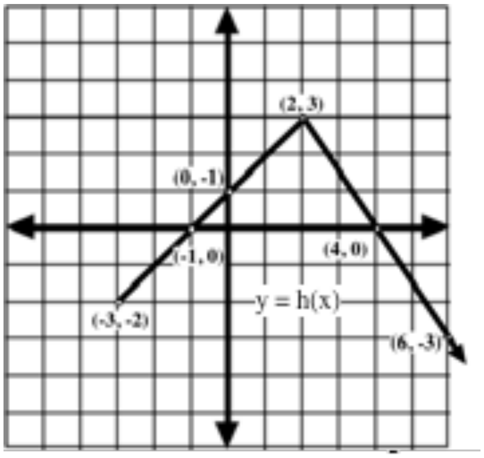
6. **Given the graph of $y = h(x)$, sketch the graph of. $y = -h(x - 1) + 3$.**



*****SEE YOUR TEACHER FOR MARKING KEY*****

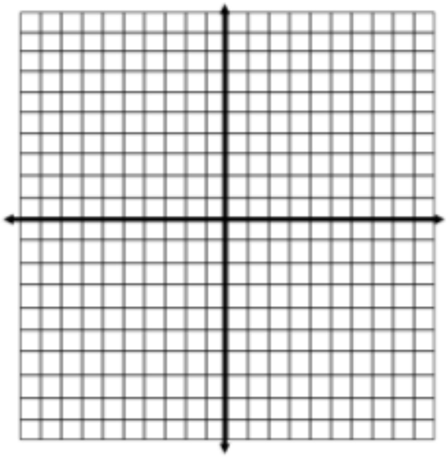
LEARNING GUIDE 2

Topic 3 Combining Transformations

EXAMPLE 1 - Sketch the function by taking the coordinates of the given points which are on the graph of $y = | x |$ and applying the appropriate transformations.

$$y = 2|(x - 4)| + 1$$

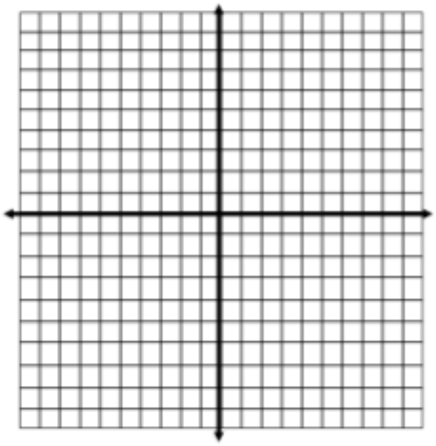
x	*y*		
0	0		
2	2		
-2	2		
4	4		
-4	4		



TRY: Sketch each function by taking the coordinates of the given points which are on the graph of y = | x | and applying the appropriate transformations.

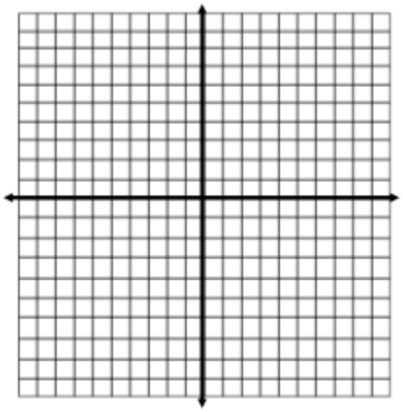
$y = | 2(x + 3) | + 1$

x	*y*		
0	0		
2	2		
-2	2		
4	4		
-4	4		



TRY: **Sketch:** $y = 3\left|\frac{1}{2}(x - 2)\right|$

x	*y*		
0	0		
2	2		
-2	2		
4	4		
-4	4		



TRY: Describe the transformation of each function below:

a) $y = 2(x + 1)^2 + 3$

c) $y = (2x - 2)^3 + 3$

b) $y = -2(x - 1)^3 + 1$

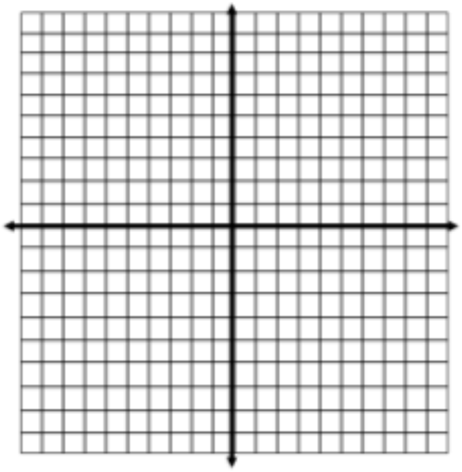
d) $y = \frac{1}{2}|2x + 3| - 1$

EXAMPLE 2 - Sketch the graph of the following function:

☆ Now, factoring is a key step in the translation of this function.

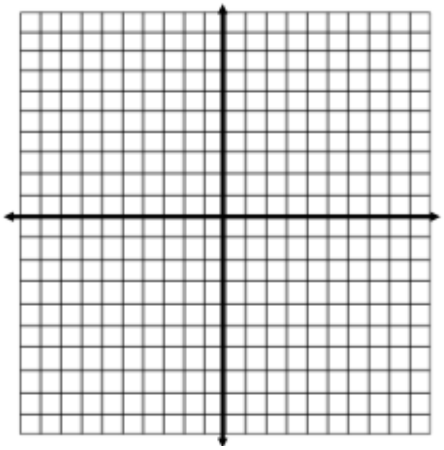
$y = \sqrt{2x - 4}$ **Describe??**

x	*y*		
0	0		
1	1		
4	2		
9	3		



TRY: Sketch the graph of the following function:

$y + 2 = 2(2x + 4)^2$



EXAMPLE 2b - Describe what happens to the graph of the function y = f(x) if $f(x) = x^2 + 1$ when you make each change to its equation. Then write the new equation:

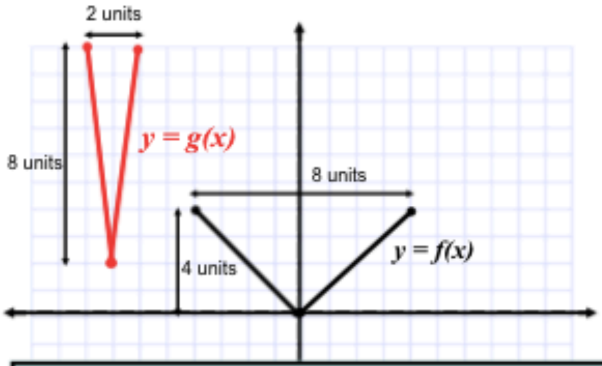
a) Replace y = f(x) with $y = 2f(4(x + 1))$.

TRY:

b) Replace y = f(x) with $y = f(2x + 2)$.

EXAMPLE 3 - Write the Equation of a Transformed Function Graph

The graph of the function $y = g(x)$ represents a transformation of the graph of $y = f(x)$. Determine the equation of $g(x)$ in the form $y = af(b(x - h)) + k$.



Solution: Locate the vertex of each function $y = f(x)$ 0,0 $y = g(x)$ -7,2 This gives you the translation 7 units to the left and 2 units up. $h = -7$ & $k = 2$

There is no reflection.

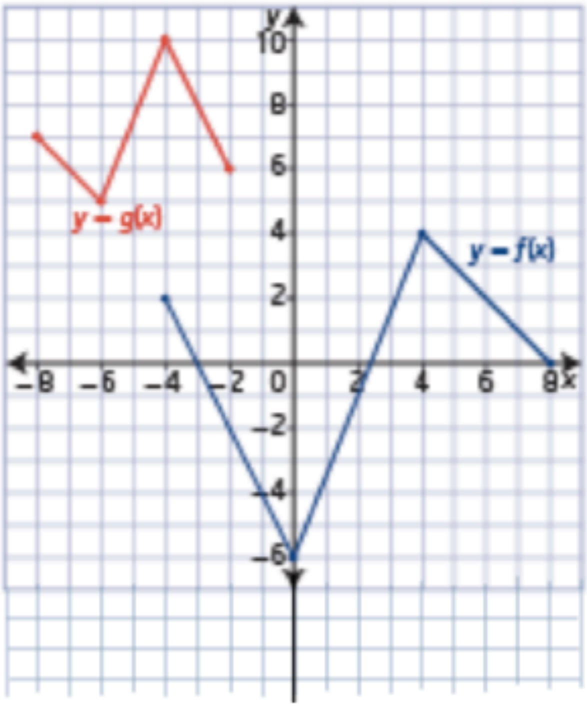
Now compare distances between key points. In the vertical direction 4 units becomes 8 units therefore you have a vertical stretch by a factor of 2. In the horizontal direction 8 units becomes 2 units, therefore there is a horizontal stretch by a factor of 1/4. This gives you an $a = 2$ & $b = 4$ → $g(x) = 2f(4(x + 7) +2$

WATCH OUT - WHEN THERE IS A REFLECTION INVOLVED!!

★ ⇨ use invariant points to find *h* and *k*.

Determine the equation of g(x) given f(x) in the form: $y = af(b(x - h)) + k$

c)



STEPS:

1. Look for reflections??

[✓ **yes**] *Both*

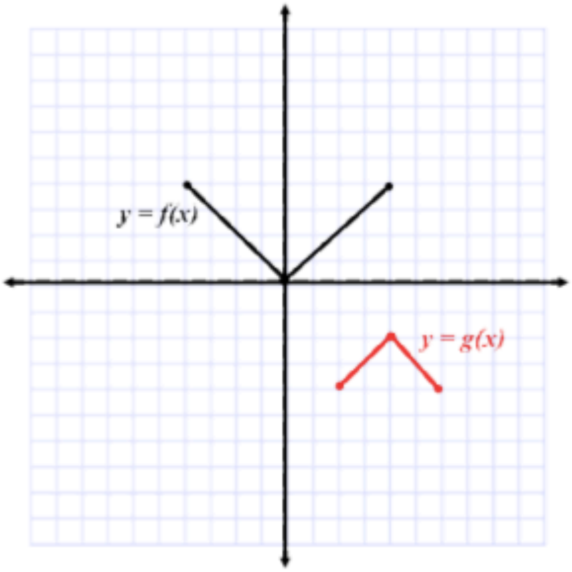
2. Find h & k

Reflection on *x*-axis ... look for *x*-int that's your invariant point ***k*** **=**
(up or down)

Reflection on *y*-axis ... look for *y*-int that's your invariant point ***h*** **=**
(right or left)

3. Find stretches

TRY: The graph of the function y = g(x) represents a transformation of the graph of y = f(x). Determine the equation of g(x) in the form y = af(b(x - h)) + k .



EXAMPLE 3b - Use the descriptions below to write a new equation for the basic function $y = x^2$.

a) Compress horizontally by a factor of $\frac{1}{2}$ and expand vertically by a factor of 5.

TRY:

b) Expand horizontally by a factor of 3, then translate 3 unit up.

Complete Assignment Questions #1 - #4

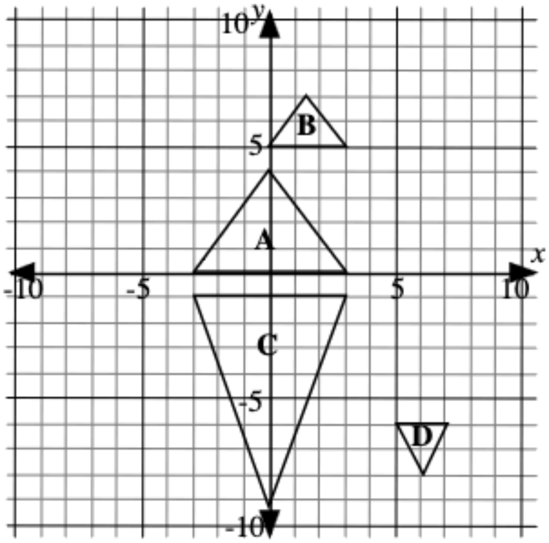
Assignment

1. Describe the series of transformations required to transform:

a) graph A to graph B.

b) graph A to graph C.

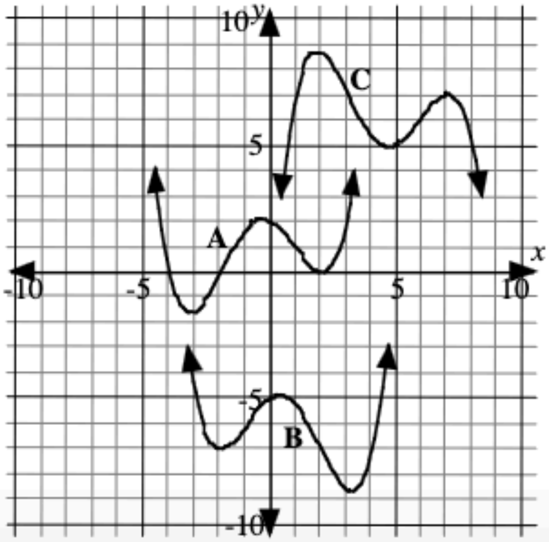
c) graph A to graph D.



2. Describe the series of transformations required to transform:

a) graph A to graph B.

b) graph A to graph C.



3. Describe which transformations are applied to a graph of a function when the following changes are made to its equation. Does the order in which the transformations are performed affect the final graph?

a) Replace x with $x + 2$ and y with $-y$.

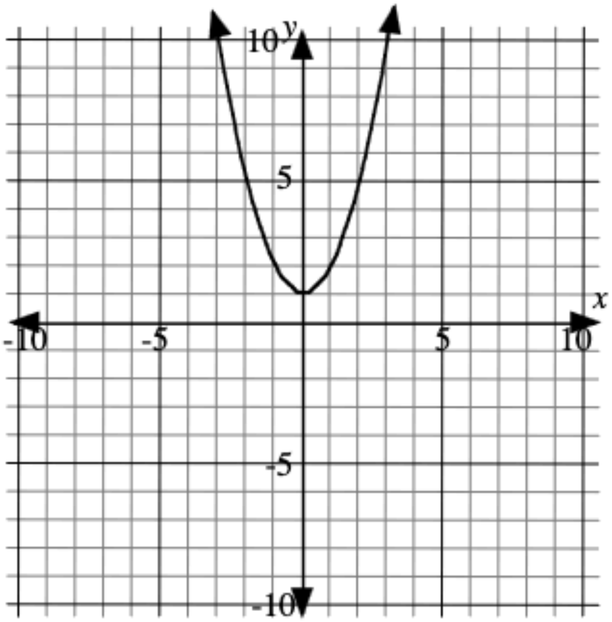
b) Replace x with $4x$ and y with $y - 7$.

c) Replace x with $\frac{1}{3}x$, y with $-2y$, and y with $y + 2$.

d) Replace x with $2x$, y with $\frac{1}{4}y$, x with $-x$, and y with $y + 10$.

4. A graph of the parabola $y = x^2 + 1$ is shown. The following transformations are applied to $y = x^2 + 1$ **in the order shown**:

- a vertical translation down 3 units
- a reflection in the x-axis
- a vertical stretch about the x-axis by a factor of $\frac{1}{4}$
- a horizontal translation to the left 2 units



a) For each transformation:

- graph the image on the grid
- write the replacement for x or y and the current equation in the table.

Transformation	***Replacement for x or y***	***Current Equation***
1 a vertical translation down 3 units		
2 a reflection in the x-axis		
3 a vertical stretch by a factor of $\frac{1}{4}$		
4 a horizontal translation left 2 units		

b) Write the equation which represents the final position of the graph.

Answer Key

1 . **a)** vertical stretch by a factor of $\frac{1}{2}$ about the x-axis, horizontal stretch by a factor of $\frac{1}{2}$ about the y-axis, then a translation of 1.5 units right, and 5 units up

b) vertical stretch by a factor of 2 about the x-axis, a reflection in the x-axis, then a vertical translation 1 unit down

c) vertical stretch by a factor of $\frac{1}{2}$ about the x-axis, horizontal stretch by a factor of $\frac{1}{3}$ about the y-axis, a reflection in the x-axis, then a translation 6 units right and 6 units down

2 . **a)** reflection in the y-axis, and a translation 7 units down

b) reflection in the x-axis, and then a translation 5 units right and 7 units up

3 . **a)** horizontal translation 2 units left and a reflection in the x-axis; no

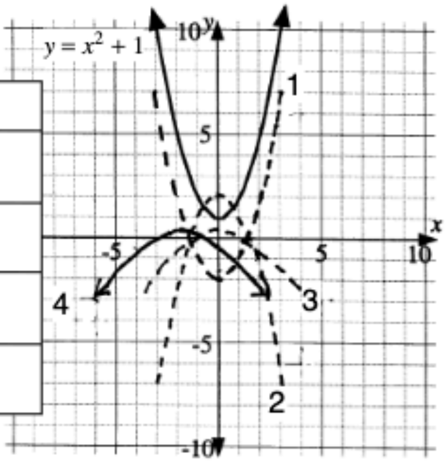
b) horizontal stretch by a factor of $\frac{1}{4}$ about the y-axis, and a vertical translation 7 units up; no

c) horizontal stretch by a factor of 3 about the y-axis, vertical stretch by a factor of $\frac{1}{2}$ about the x-axis, reflection in the x-axis and a vertical translation 2 units down; yes

d) horizontal stretch by a factor of $\frac{1}{2}$ about the y-axis, vertical stretch by a factor of 4 about the x-axis, a reflection in the y-axis and a vertical translation 10 units down; yes

4 . **a)**

Transformation	*Replacement for x or y*	*Current Equation*
1 a vertical translation down 3 units	replace y with $y + 3$	$y + 3 = x^2 + 1$ $y = x^2 - 2$
2 a reflection in the x-axis	replace y with $-y$	$-y = x^2 - 2$ $y = -x^2 + 2$
3 a vertical stretch of a factor of $\frac{1}{4}$	replace y with $4y$	$4y = -x^2 + 2$ $y = -\frac{1}{4}x^2 + \frac{1}{2}$
4 a horizontal translation left 2 units	replace x with $x + 2$	$y = -\frac{1}{4}(x + 2)^2 + \frac{1}{2}$



b) $y = -\frac{1}{4}(x + 2)^2 + \frac{1}{2}$

Complete Assignment Questions #1 - #6

Assignment

1. Describe how the graph of $y = f(x)$ can be transformed to the graph of

a) $y = f[2(x-1)] + 5$

b) $y = 2f(x+4) - 5$

c) $y = f\left(\frac{1}{2}x + 6\right) + 1$

2. Consider the function $y = f(x)$. In each case determine:

- the replacements for x and y which would result in the following combinations of transformations
- the equation of the transformed function in the form $y = af[b(x-h)] + k$

a) a horizontal stretch by a factor of 3 about the y-axis and a vertical translation of 6 units up.

b) a reflection in the y-axis, a horizontal translation of 3 units right, and vertical translation of 5 units down.

c) a horizontal stretch by a factor $\frac{2}{3}$ about the y-axis, a vertical stretch by a factor of $\frac{2}{5}$ about the x-axis, a reflection in the x-axis, and a vertical translation of 1 unit up.

3. Describe how the graph of the second function compares to the graph of the first function.

a) $y = x^4$, $\quad -4y = (x - 2)^4$

b) $y = |x|$, $\quad y = \left|\frac{1}{3}(x + 2)\right|$

c) $y = \sqrt{x}$, $\quad y - 1 = 2\sqrt{4x - 8}$

4. In each case the combination of transformations are applied in the order given to transform the graph of $y = f(x)$ to the graph of $y = af[b(x - h)] + k$.
Determine the values of a, b, h, and k.

a) a horizontal stretch by a factor of $\frac{3}{5}$ about the y-axis and a reflection in the x-axis.

b) a vertical stretch by a factor of $\frac{1}{3}$ about the x-axis and a reflection in the y-axis.

c) a vertical stretch by a factor of 2 about the x-axis, then a translation 5 units to the left and 2 units up.

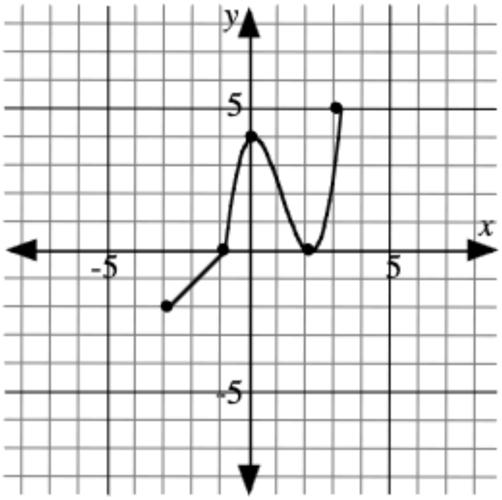
d) a horizontal stretch by a factor of 4 about the y-axis, a vertical stretch by a factor of 2 about the x-axis, a reflection in the y-axis and then a translation of 10 units down.

e) a translation of 6 units right, then a horizontal stretch by a factor of $\frac{1}{2}$ about the y-axis and a reflection in the x-axis.

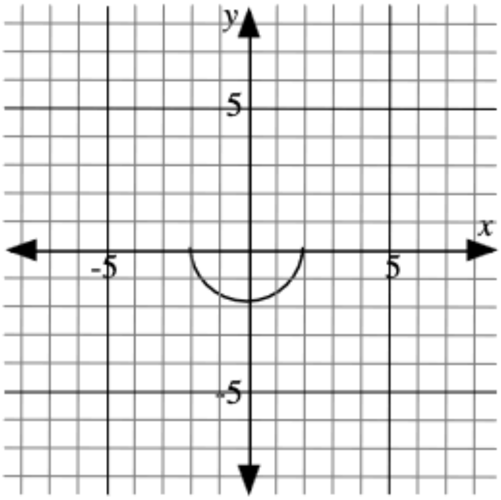
5. The function $f(x) = \sqrt{x}$ is transformed into the function $g(x)$ by stretching horizontally by a factor of 6 about the y-axis, stretching vertically by a factor of 3 about the x-axis, reflecting in the x-axis, and translating 1 unit up and $\frac{1}{2}$ unit to the right.
Write the equation for $g(x)$.

6. The graph of $y = f(x)$ is shown. Sketch the graph of:

a) $y + 4 = -\frac{1}{2}f(x + 2)$



b) $y = -4f\left(\frac{1}{2}x + 1\right)$



Answer Key

1. **a)** horizontal stretch by a factor of $\frac{1}{2}$ about the y-axis, then a translation 1 unit right and 5 units up

b) vertical stretch by a factor of 2 about the x-axis, then a translation 4 units left and 5 units down

c) horizontal stretch by a factor of 2 about the y-axis, then a translation 12 units left and 1 unit up

2. **a)** replace x with $\frac{1}{3}x$ and y with $y - 6$ $\quad y = f\left(\frac{1}{3}x\right) + 6$

b) replace x with $-x$, x with $x - 3$, and y with $y + 5$ $\quad y = f(-(x - 3)) - 5$ or $y = f(-x + 3) - 5$

c) replace x with $\frac{3}{2}x$, y with $\frac{5}{2}y$, y with $-y$, and y with $y - 1$ $\quad y = -\frac{2}{5}f\left(\frac{3}{2}x\right) + 1$

3. **a)** vertical stretch by a factor of $\frac{1}{4}$ about the x-axis, reflection in the x-axis, and a horizontal translation 2 units right

b) horizontal stretch by a factor of 3 about the y-axis, then a horizontal translation 2 units left

c) vertical stretch by a factor of 2 about the x-axis, horizontal stretch by a factor of $\frac{1}{4}$ about the y-axis, then a translation 2 units right and 1 unit up

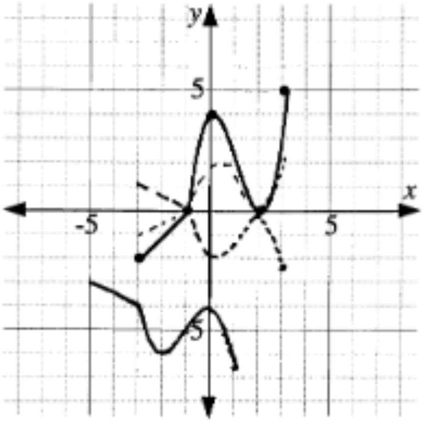
4. **a)** $a = -1$ $\quad b = \frac{5}{3}$ $\quad h = 0$ $\quad k = 0$ $\qquad$ **b)** $a = \frac{1}{3}$ $\quad b = -1$ $\quad h = 0$ $\quad k = 0$

c) $a = 2$ $\quad b = 1$ $\quad h = -5$ $\quad k = 2$ $\qquad$ **d)** $a = 2$ $\quad b = -\frac{1}{4}$ $\quad h = 0$ $\quad k = -10$

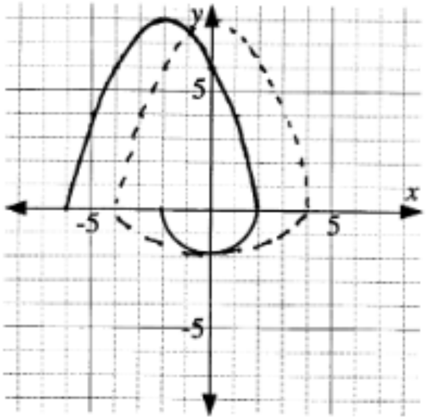
e) $a = -1$ $\quad b = 2$ $\quad h = 3$ $\quad k = 0$

5. $g(x) = -3\sqrt{\frac{1}{6}\left(x - \frac{1}{2}\right)} + 1$

6. **a)**

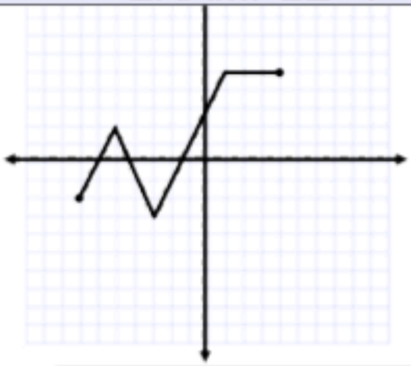


b)



Topic 4 Inverse of a Relation

EXAMPLE 1 - Graph the Inverse

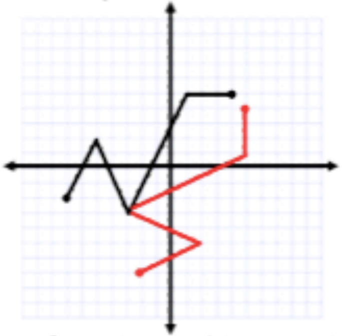


a) Sketch the graph of the inverse.
b) State the Domain and Range.
c) Determine whether the relation
and its inverse are functions.

Solution: Take key points from the relation and then interchange the x & y coordinates.

Points of the Relation	Points of the Inverse Relation
(-7, -2)	(-2, -7)
(-5, 2)	(2, -5)
(-3, -3)	(-3, -3)
(1, 5)	(5, 1)
(4, 5)	(5, 4)

a)



b)

	Domain	Range
Relation	$-7 \le x \le 4$	$-3 \le y \le 5$
Inverse Relation	$-3 \le x \le 5$	$-7 \le y \le 4$

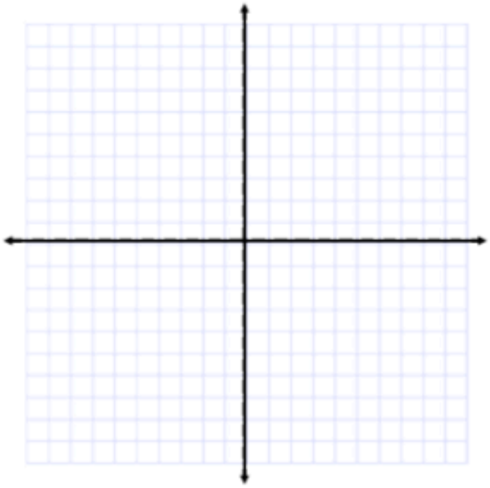
c) Relation is a function and inverse is NOT a function. Did not pass the Horizontal line TEST of the original function, THEREFORE, does not pass the Vertical line TEST of the inverse function.

TRY: Sketch the inverse of the following function:

$f(x) = x^2 + 1$

f(x)

x	y	x	y



EXAMPLE 2 - How to put a restriction on the domain of $f(x)$ so that it's inverse is a function.

$f(x) = x^2 + 1$

$y = x^2 + 1$

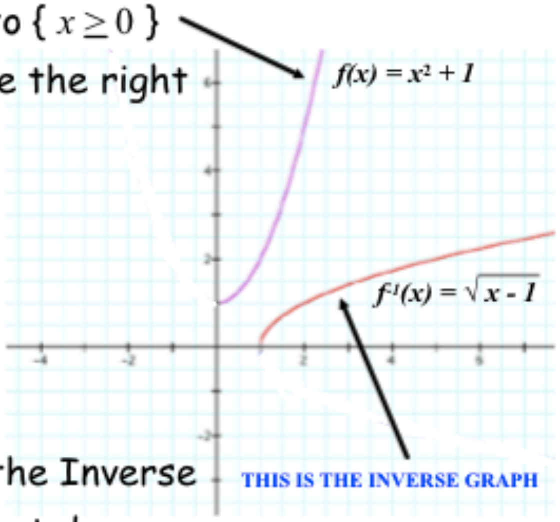
$x = y^2 + 1$

$x - 1 = y^2$

$\sqrt{x-1} = \sqrt{y^2}$

$y = +\sqrt{x-1} \quad \therefore \quad f^{-1}(x) = +\sqrt{x-1}$

Restrict the Domain to $\{ x \geq 0 \}$ therefore you only see the right side of the parabola.



With the restriction on the Domain the Inverse function will only have the positive part drawn.

As you can now see it is a Function. [passed the vertical line test]

EXAMPLE 3 - Determine the Equation of the Inverse

★**In** Inverse Functions the x and y interchange with each other.

Algebraically determine the inverse equation of $f(x) = 3x + 2$

1st - replace $f(x)$ with a y $\quad y = 3x + 2$

2nd - interchange x with y $\quad x = 3y + 2$

3rd - solve for y $\quad x - 2 = 3y \quad$ *subtract 2 from each side*

$$\frac{x-2}{3} = \frac{3y}{3} \quad \textit{divide by 3 each side}$$

$$y = \frac{x-2}{3} \quad \textit{rewrite as} \quad f^{-1}(x) = \frac{x-2}{3}$$

inverse symbol

TRY: Algebraically determine the inverse equation of $f(x) = \frac{x+8}{3}$

TRY: Algebraically determine the inverse equation of $f(x)=\frac{-2x}{x+3}$

TRY: Algebraically determine the inverse equation of

$$f(x)=2(x-4)^2+1$$

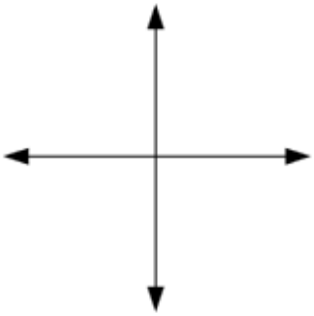
★ **Determine a restriction on the domain so that the inverse is a function.**

Complete Assignment Questions #4 - #6

4. **a)** Graph the function $f(x) = x^2 + 4$.

b) Graph the inverse of $f(x)$.

c) Find the equation of the inverse function in the form $x = f(y)$ and solve for y.



5. A function f is defined by $y = 2x + 1$. Which of the following is the equation of the inverse of f?

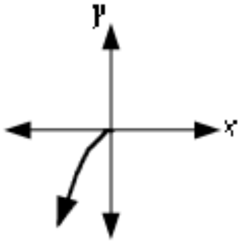
A. $x = \frac{y-1}{2}$

B. $x = 2y + 1$

C. $y = \frac{x+1}{2}$

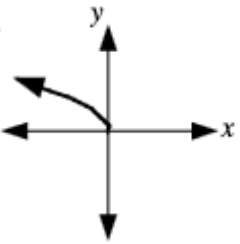
D. $y = 2x - 1$

6. The graph of the function $y = f(x)$ is shown in the diagram below.

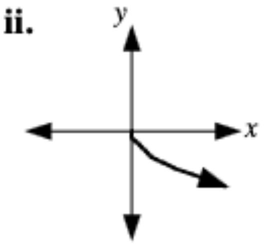


Which of the following represents $f^{-1}(x)$?

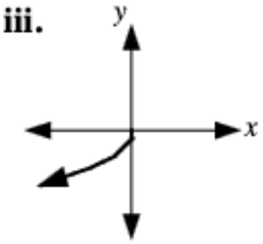
i.



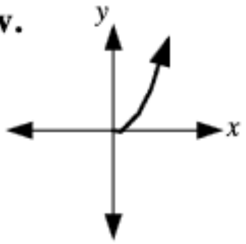
ii.



iii.



iv.



A. i
B. ii
C. iii
D. iv

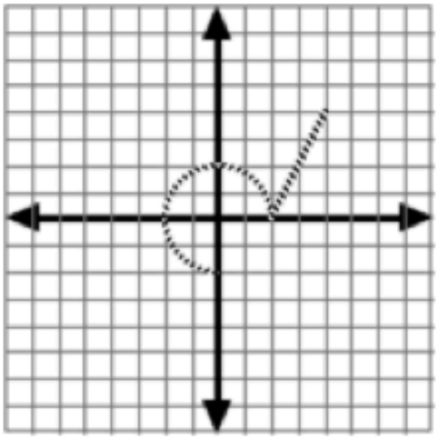
Answer Key

4. c) $x = y^2 + 4,\ y = \pm\sqrt{x-4}$ **5.** B **6.** C

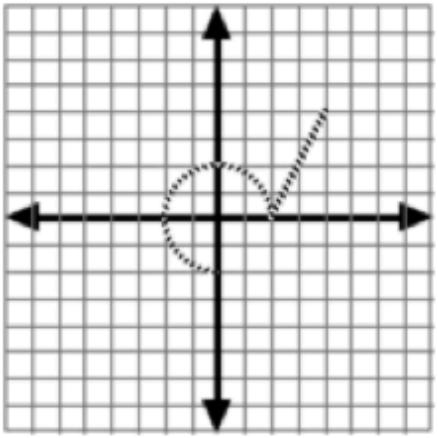
LG 2 Worksheet B (Combining Transformations)

1. Given the dotted $y = f(x)$ below:

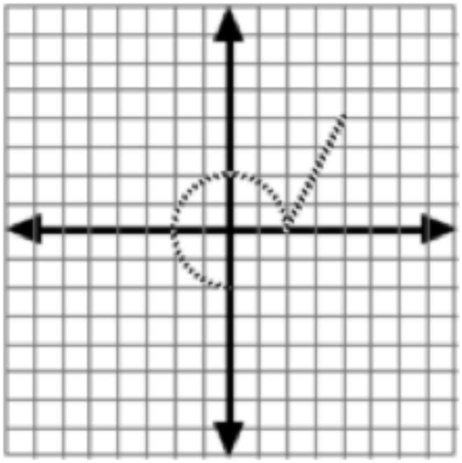
a) Find $y = 2f(x+3) - 4$



b) Find $y = -f(-x)$

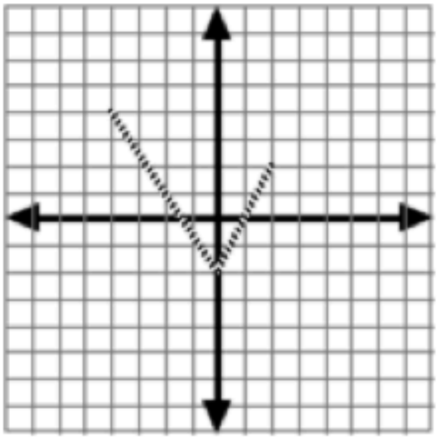


c) Find $y = f(2x+6) - 1$

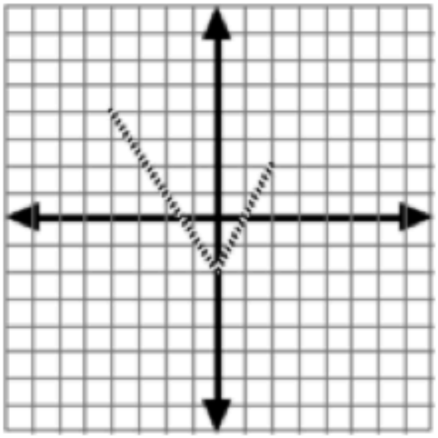


2. Given the dotted $y = f(x)$ below:

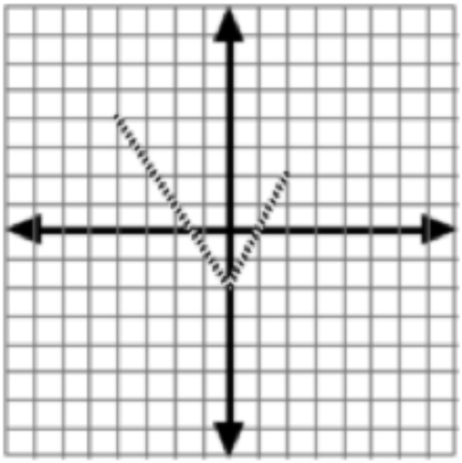
a) Find $y - 3 = \frac{1}{2}f\big(2(x-2)\big)$



b) Find $-y = f(2x-6)$



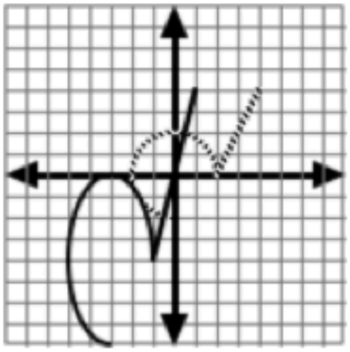
c) Find $x = f(y)$ *or* $y = f^{-1}(x)$



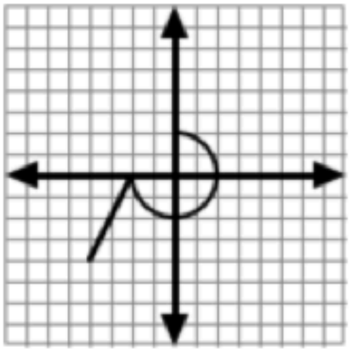
Answer Key

Answer Key

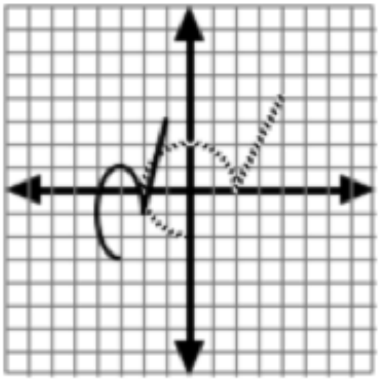
1. a.



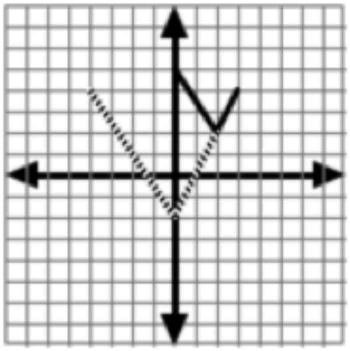
b.



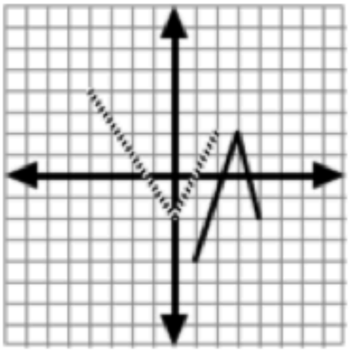
c.



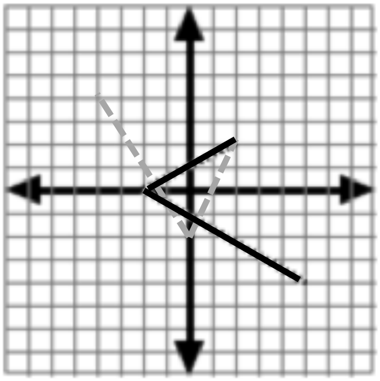
2. a.



b.



c.



LG 2 Worksheet C (Describing Transformations)

What is the effect on the graph of $y = f(x)$ if it is transformed to each of the following?

1. $y = 2f(3x - 12) + 6$

2. $y = -3f(2x - 10) - 8$

3. $y + 6 = 5f(\frac{1}{2}x - 4)$

4. $y - 5 = -3f(8 - 4x)$

5. $2y + 6 = f(4x - 20)$

6. $4y + 12 = -f(5 - \frac{1}{3}x)$

Given the following effects, create an equation that is a transformation on $y = f(x)$.

1. Vertically expanded by a factor of 3
Vertically translated down 9
Horizontally compressed by a factor of $\frac{1}{2}$
Horizontally translated left 4

2. Vertically compressed by a factor of $\frac{1}{3}$
Vertically translated up 6
Horizontally expanded by a factor of 2
Horizontally translated right 2

3. Vertically expanded by a factor of 2
Vertically translated down 4
Reflection in the x-axis
Horizontally expanded by a factor of 4
Horizontally translated right 8

4. Vertically compressed by a factor of $\frac{1}{4}$
Vertically translated up 2
Horizontally compressed by a factor of $\frac{1}{3}$
Horizontally translated right 3

5. Vertically expanded by a factor of 2
Vertically translated up 2
Horizontally expanded by a factor of 6
Horizontally translated left 12
Reflection in the y-axis

6. Vertically expanded by a factor of 3
Vertically translated up 6
Reflection in the x-axis
Horizontally compressed by a factor of $\frac{1}{6}$
Horizontally translated right 2
Reflection in the y-axis

Answer Key

1. Vert. exp. by a factor of 2
Trans. up 6
Horiz. comp. by a factor of $\frac{1}{3}$
Trans. right 4

2. Vert. exp. by a factor of 3
Trans. down 8
Refl. in x-axis
Horiz. comp. by a factor of $\frac{1}{2}$
Trans. right 5

3. Vert. exp. by a factor of 5
Trans. down 6
Horiz. exp. by a factor of 2
Trans. right 8

4. Vert. exp. by a factor of 3
Trans. up 5
Refl. in x-axis
Horiz. comp. by a factor of $\frac{1}{4}$
Trans. right 2
Refl. in y-axis

5. Vert. comp. by a factor of $\frac{1}{2}$
Trans. down 3
Horiz. comp. by a factor of $\frac{1}{4}$
Trans. right 5

6. Vert. exp. by a factor of $\frac{1}{4}$
Trans. down 3
Refl. in x-axis
Horiz. exp. by a factor of 3
Trans. right 15
Refl. in y-axis

Answer Key

1. $y = 3f(2x+8)-9$
or
$\frac{1}{3}y + 3 = f(2x+8)$

2. $y = \frac{1}{3}f(\frac{1}{2}x-1)+6$
or
$3y-18 = f(\frac{1}{2}x-1)$

3. $y = -2f(\frac{1}{4}x-2)-4$
or
$-\frac{1}{2}y-2 = f(\frac{1}{4}x-2)$

4. $y = \frac{1}{4}f(3x-9)+2$
or
$4y-8 = f(3x-9)$

5. $y = 2f(-\frac{1}{6}x-2)+2$
or
$\frac{1}{2}y-1 = f(-\frac{1}{6}x-2)$

6. $y = -3f(-6x+12)+6$
or
$-\frac{1}{3}y+2 = f(-6x+12)$

PRE-CALCULUS 12

Seminar Notes

Learning Guides 3

POLYNOMIAL FUNCTIONS

Many common functions are polynomial functions. In this unit we describe polynomial functions and look at some of their properties.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

- recognise when a rule describes a polynomial function, and write down the degree of the polynomial,
- recognize the typical shapes of the graphs of polynomials, of degree up to 5,
- understand what is meant by the multiplicity of a root of a polynomial,
- sketch the graph of a polynomial, given its expression as a product of linear factors

Characteristics of Polynomial Functions

A polynomial function has one variable, x, and has an exponent n, where n is a whole number. The coefficient of the greatest power of x is the leading coefficient.

Examples:

$f(x) = 2x + 3$	**characteristics**	$y = x^3 + 2x^2 - 5x - 7$
1	the degree	3
2	the leading coefficient	1
3	the constant term	-7

Topic 1 Identify Polynomial Functions

EXAMPLE 1 - Which functions are polynomials? Justify your answer. State the degree, the leading coefficient, and the constant term of each polynomial function.

a) $g(x) = \sqrt{x} + 5$

b) $f(x) = 3x^2$

c) $y = |x|$

$y = -2x^3 + 2x^2 - 6x - 1$

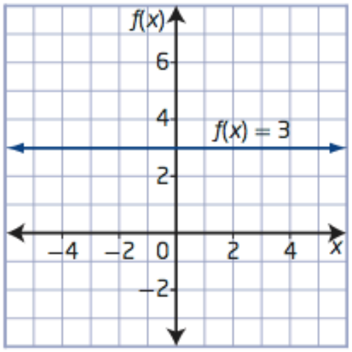
Let’s sketch the different order of polynomials

Types of Polynomials	$y = x$	$y = x^2$	$y = x^3$	$y = x^4$	$y = x^5$
Positive *look* →					
Domain					
Range					
Negative *look* →					
Domain					
Range					

Degree 0: Constant Function

Even degree

Number of x-intercepts: 0 (for $f(x) \neq 0$)



Example: $f(x) = 3$

End behaviour: extends horizontally

Domain: $\{x \mid x \in R\}$

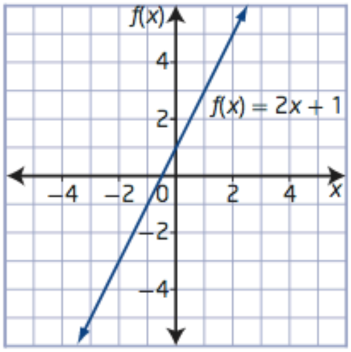
Range: $\{3\}$

Number of x-intercepts: 0

Degree 1: Linear Function

Odd degree

Number of x-intercepts: 1



Example: $f(x) = 2x + 1$

End behaviour: line extends down into quadrant III and up into quadrant I

Domain: $\{x \mid x \in R\}$

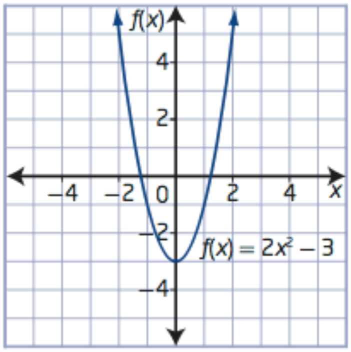
Range: $\{y \mid y \in R\}$

Number of x-intercepts: 1

Degree 2: Quadratic Function

Even degree

Number of x-intercepts: 0, 1, or 2



Example: $f(x) = 2x^2 - 3$

End behaviour: curve extends up into quadrant II and up into quadrant I

Domain: $\{x \mid x \in R\}$

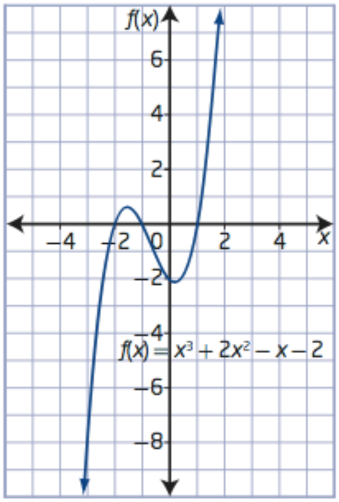
Range: $\{y \mid y \geq -2, y \in R\}$

Number of x-intercepts: 2

Degree 3: Cubic Function

Odd degree

Number of x-intercepts: 1, 2, or 3



Example:

$f(x) = x^3 + 2x^2 - x - 2$

End behaviour: curve extends down into quadrant III and up into quadrant I

Domain: $\{x \mid x \in R\}$

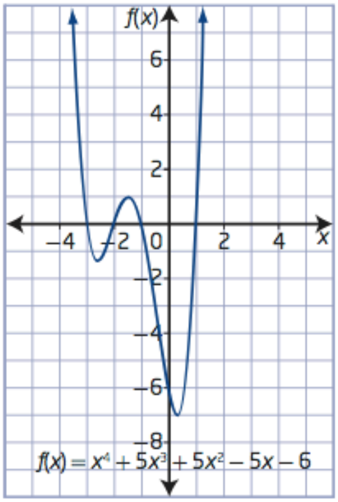
Range: $\{y \mid y \in R\}$

Number of x-intercepts: 3

Degree 4: Quartic Function

Even degree

Number of x-intercepts: 0, 1, 2, 3, or 4



Example:

$f(x) = x^4 + 5x^3 + 5x^2 - 5x - 6$

End behaviour: curve extends up into quadrant II and up into quadrant I

Domain: $\{x \mid x \in R\}$

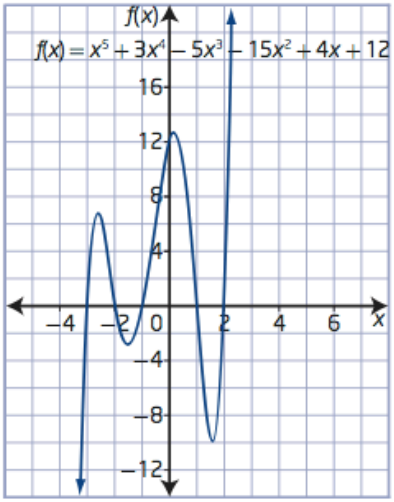
Range: $\{y \mid y \geq -6.91, y \in R\}$

Number of x-intercepts: 4

Degree 5: Quintic Function

Odd degree

Number of x-intercepts: 1, 2, 3, 4, or 5



Example:

$f(x) = x^5 + 3x^4 - 5x^3 - 15x^2 + 4x + 12$

End behaviour: curve extends down into quadrant III and up into quadrant I

Domain: $\{x \mid x \in R\}$

Range: $\{y \mid y \in R\}$

Number of x-intercepts: 5

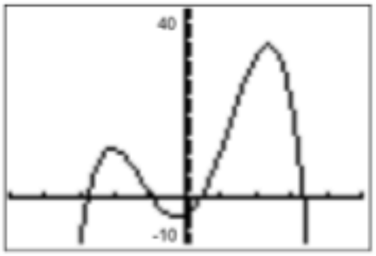
EXAMPLE 2 - Match a Polynomial Function with its Graph

____ $f(x) = x^3 + x^2 - 5x + 3$

____ $g(x) = x^4 + 4x^3 - x^2 - 16x - 12$

____ $h(x) = -x^4 + 10x^2 + 5x - 4$

A



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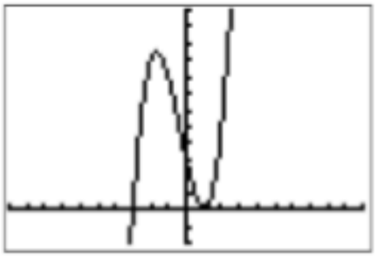

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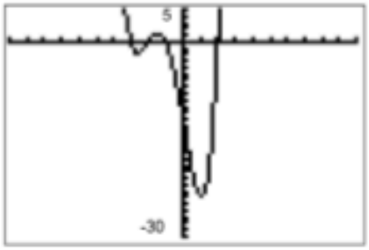


Complete Assignment Questions #1 - #3

Assignment

1. Identify whether each of the following is a polynomial function. Justify your answers.

a) $h(x) = 2 - \sqrt{x}$

b) $y = 3x + 1$

c) $f(x) = 3^x$

2. What are the degree, type, leading coefficient, and constant term of each polynomial function?

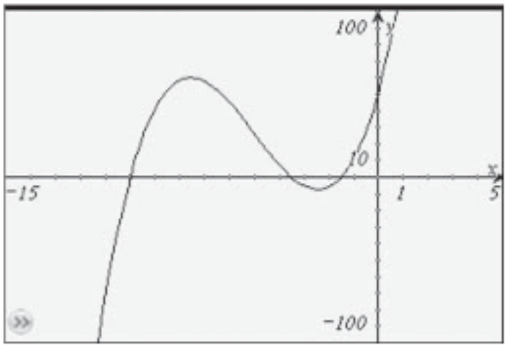
a) $f(x) = -x + 3$

b) $y = 9x^2$

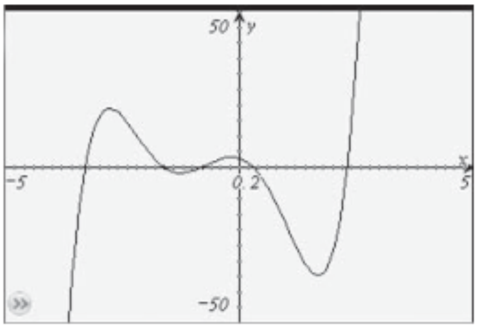
c) $g(x) = 3x^4 + 3x^2 - 2x + 1$

3. For each of the following:
- determine whether the graph represents an odd-degree or an even-degree polynomial function
- determine whether the leading coefficient of the corresponding function is positive or negative
- state the number of x-intercepts
- state the domain and range

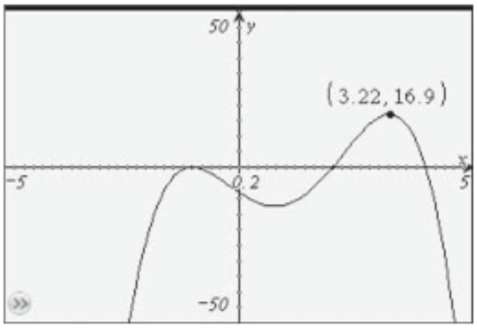
a)



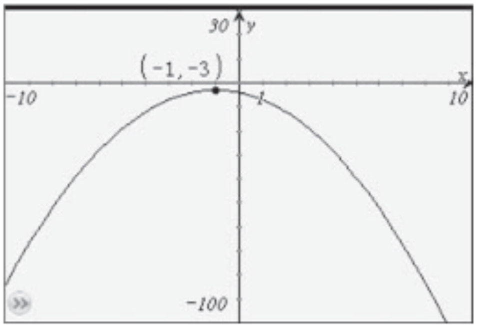
b)



c)



d)



Answer Key

1. a) No, this is a square root function.
b) Yes, this is a polynomial function of degree 1.
c) No, this is an exponential function.

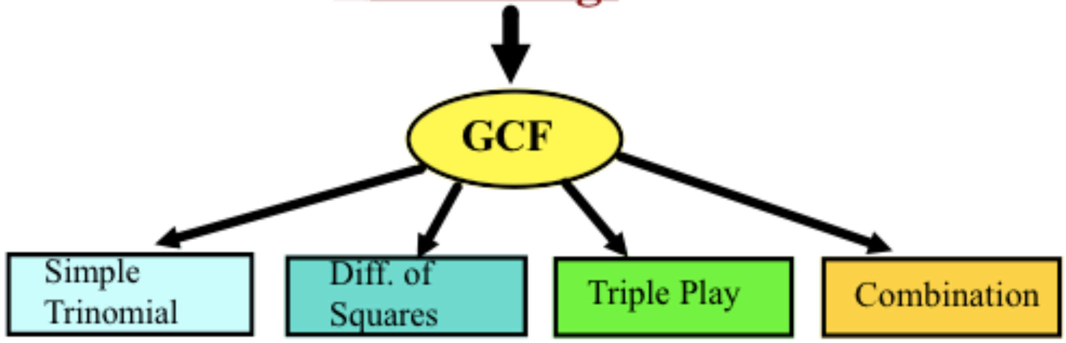
2. a) degree 1, linear, −1, 3
b) degree 2, quadratic, 9, 0
c) degree 4, quartic, 3, 1

3. a) odd degree, positive leading coefficient, 3 x-intercepts, domain $\{x \mid x \in \mathrm{R}\}$ and range $\{y \mid y \in \mathrm{R}\}$
b) odd degree, positive leading coefficient, 5 x-intercepts, domain $\{x \mid x \in \mathrm{R}\}$ and range $\{y \mid y \in \mathrm{R}\}$
c) even degree, negative leading coefficient, 3 x-intercepts, domain $\{x \mid x \in \mathrm{R}\}$ and range $\{y \mid y \leq 16.9, y \in \mathrm{R}\}$
d) even degree, negative leading coefficient, 0 x-intercepts, domain $\{x \mid x \in \mathrm{R}\}$ and range $\{y \mid y \leq -3, y \in \mathrm{R}\}$

REVIEW

of

Factoring



Examples:

Factoring Practice

Factor each completely.

1) $b^2 + 8b + 7$

2) $n^2 - 11n + 10$

3) $m^2 + m - 90$

4) $n^2 + 4n - 12$

5) $n^2 - 10n + 9$

6) $b^2 + 16b + 64$

Factor each completely.

1) $3p^2 - 2p - 5$

2) $2n^2 + 3n - 9$

3) $3n^2 - 8n + 4$

4) $5n^2 + 19n + 12$

Factor each completely.

1) $16n^2 - 9$

2) $4m^2 - 25$

3) $16b^2 - 40b + 25$

4) $4x^2 - 4x + 1$

5) $9x^2 - 1$

6) $n^2 - 25$

15) $2n^2 + 6n - 108$

16) $5n^2 + 10n + 20$

Divide a Polynomial by a Binomial of the Form x - a

Example: $2x^3 + x^2 - 5x + 4 \div x + 2$

Long Division **Synthetic Division**

Topic 2 Remainder Theorem

EXAMPLE 1 - Divide a Polynomial by a Binomial

The result of the division of a polynomial in x, $P(x)$, by a binomial of the form $x - a$, $a \in I$, is $\frac{P(x)}{x-a} = Q(x) + \frac{R}{x-a}$, where $Q(x)$ is the quotient and R is the remainder.

a) Divide the polynomial $P(x) = 5x^3 + 10x - 13x^2 - 9$ by $x - 2$. Express the result in the form $\frac{P(x)}{x-a} = Q(x) + \frac{R}{x-a}$.

b) Identify any restrictions on the variable.

TRY: *Divide the polynomial* $P(x) = x^4 - 2x^3 + x^2 - 3x + 4$ *by* $x - 1$. *Express the result in the form* $\frac{P(x)}{x-a} = Q(x) + \frac{R}{x-a}$. *Identify any restrictions.*

EXAMPLE 2 - Divide a Polynomial Using Synthetic Division

$$(x^2 + \underset{\substack{\uparrow \\ \textbf{Polynomial}}}{4}x - 5) \div (x \underset{\substack{\uparrow \\ \textbf{Divisor}}}{-} 1)$$

TRY: *Use synthetic division :* $\dfrac{x^3 + 7x^2 - 3x + 4}{x - 2}$.

Example of using Synthetic Division where a 0 must be inserted.

$x^4 + 2x^3 - 6x + 1 \ \div x - 1$

EXAMPLE 3 - Apply the Remainder Theorem

Use the remainder theorem to determine the remainder when

$P(x) = x^2 - 10x + 6$ *is divided by* $x + 4$.

TRY: *What is the remainder when* $11x - 4x^4 - 7$ *is divided by* $x - 3$?

EXAMPLE 4b - Apply the Remainder Theorem to find the Value of k

For each dividend, determine the value of k if the remainder is 5.

a) $(x^3 + 2x^2 - kx - 1) \div (x + 1)$

b) $(x^3 - kx^2 + 7x + 3) \div (x - 2)$

Complete Assignment Questions #4 - #9

Assignment

4. Use the degree and the sign of the leading coefficient of each function to describe the end behaviour of the corresponding graph. State the possible number of x-intercepts and the value of the y-intercept.

a) $f(x) = x^2 + 3x - 1$

b) $g(x) = -4x^3 + 2x^2 - x + 5$

c) $h(x) = -7x^4 + 2x^3 - 3x^2 + 6x + 4$

4. Determine each quotient, Q, using synthetic division.

a) $(x^3 + x^2 + 3) \div (x + 4)$

b) $\dfrac{m^4 - 2m^3 + m^2 + 12m - 6}{m - 2}$

c) $(2 - x + x^2 - x^3 - x^4) \div (x + 2)$

5. Perform each division. Express the result in the form $\frac{P(x)}{x - a} = Q(x) + \frac{R}{x - a}$. Identify any restrictions on the variable.

a) $(x^3 + 7x^2 - 3x + 4) \div (x + 2)$

b) $\dfrac{11t - 4t^4 - 7}{t - 3}$

c) $(x^3 + 3x^2 - 2x + 5) \div (x + 1)$

6. Use the remainder theorem to determine the remainder when each polynomial is divided by $x + 2$.

a) $x^3 + 3x^2 - 5x + 2$

b) $2x^4 - 2x^3 + 5x$

7. Determine the remainder resulting from each division.

a) $(x^3 + 2x^2 - 3x + 9) \div (x + 3)$

b) $\dfrac{2t - 4t^3 - 3t^2}{t - 2}$

8. For each dividend, determine the value of k if the remainder is 3.

a) $(x^3 + 4x^2 - x + k) \div (x - 1)$

b) $(x^3 + x^2 + kx - 15) \div (x - 2)$

9. For what value of c will the polynomial $P(x) = -2x^3 + cx^2 - 5x + 2$ have the same remainder when it is divided by $x - 2$ and by $x + 1$?

Answer Key

4. a) $Q(x) = x^2 - 3x + 12$ **b)** $Q(m) = m^3 + m + 14$

c) $Q(x) = -x^3 + x^2 - x + 1$

5. a) $\dfrac{x^3 + 7x^2 - 3x + 4}{x + 2} = x^2 + 5x - 13 + \dfrac{30}{x + 2}, x \neq -2$

b) $\dfrac{11t - 4t^4 - 7}{t - 3}$

$= -4t^3 - 12t^2 - 36t - 97 - \dfrac{298}{t - 3}, t \neq 3$

c) $\dfrac{x^3 + 3x^2 - 2x + 5}{x + 1} = x^2 + 2x - 4 + \dfrac{9}{x + 1}, x \neq -1$

6. a) 16 **b)** 38 **7. a)** 9 **b)** −40

8. a) −1 **b)** 3 **9.** 11

Topic 3 The Factor Theorem

EXAMPLE 1 - Use the Factor Theorem to Test for Factors of a Polynomial

Which binomials are factors of the polynomial $P(x) = x^3 - 3x^2 - x + 3$?

a) $x - 1$ *b)* $x + 1$ *c)* $x + 3$ *d)* $x - 3$

EXAMPLE 2 - Factor Using the Integral Zero Theorem

Factor $2x^3 - 5x^2 - 4x + 3$ fully.

TRY: Factor fully $P(x) = x^3 - 4x^2 - 11x + 30$.

EXAMPLE 3 - Factor Higher-Degree Polynomials

Fully factor $x^4 - 5x^3 + 2x^2 + 20x - 24$ fully.

Complete Assignment Questions #3 - #5

Assignment

3. State whether each polynomial has $x + 2$ as a factor.

a) $5x^2 + 2x + 6$

b) $2x^3 - x^2 - 5x - 8$

c) $2x^3 + 2x^2 - x - 6$

4. What are the possible integral zeros of each polynomial?

a) $P(x) = x^3 + 3x^2 - 6x - 8$

b) $P(s) = s^3 + 4s^2 - 15s - 18$

c) $P(n) = n^3 - 3n^2 - 10n + 24$

5. Factor fully.

a) $P(x) = x^3 - 6x^2 + 11x - 6$

b) $P(x) = x^3 + 2x^2 - x - 2$

c) $P(v) = v^3 + v^2 - 16v - 16$

d) $P(x) = x^4 + 4x^3 - 7x^2 - 34x - 24$

a)	b)
c)	d)

Answer Key

3. **a)** No **b)** No **c)** No

4. **a)** ±1, ±2, ±4, ±8 **b)** ±1, ±2, ±3, ±6, ±9, ±18
   c) ±1, ±2, ±3, ±4, ±6, ±8, ±12, ±24

5. **a)** $(x-1)(x-2)(x-3)$ **b)** $(x-1)(x+1)(x+2)$
   c) $(v-4)(v+4)(v+1)$
   d) $(x+4)(x+2)(x-3)(x+1)$

Factor by Grouping

1) $12x^3-8x^2-3x+2$

2) $8x^3-4x^2-2x+1$

Rational Zero Theorem

1) $\overset{q}{12}x^3-8x^2-3x+\overset{p}{2} \longrightarrow \frac{p}{q}$

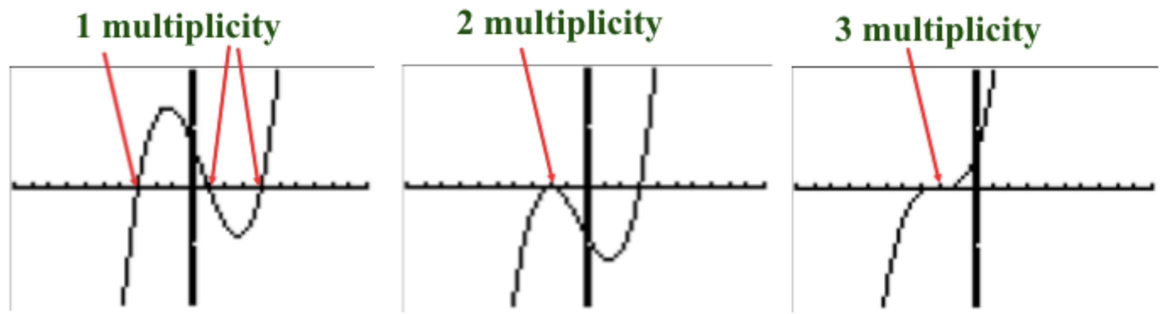
Find Factors of

p	q
±1	±1
±2	±2
	±3

⋮

Multiplicity (of a zero)

☛ the number of times a zero of a polynomial functions occurs



Topic 4 Equations & Graphs of Polynomial Functions

EXAMPLE 1 - Analyse Graphs of Polynomial Functions

For each graph of a polynomial function, determine

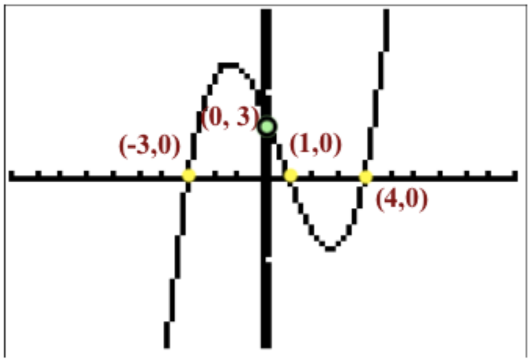
a) the degree

b) the sign of the leading coefficient & what is the leading coef.?

c) x-intercepts and the factors

d) the intervals where the function is positive or negative

a)



EXAMPLE 1b - Analyse Graphs of Polynomial Functions

For each graph of a polynomial function, determine

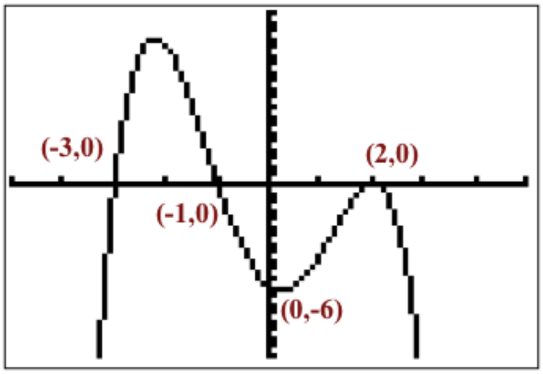
a) the degree

b) the sign of the leading coefficient & what is the leading coef.?

c) x-intercepts and the factors

d) the intervals where the function is positive or negative

b)



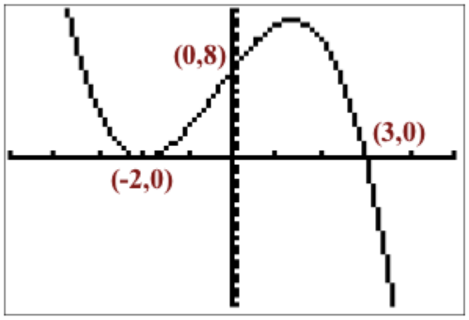
TRY: For the graph of a polynomial function, determine

a) the degree

b) the sign of the leading coefficient & what is the leading coef.?

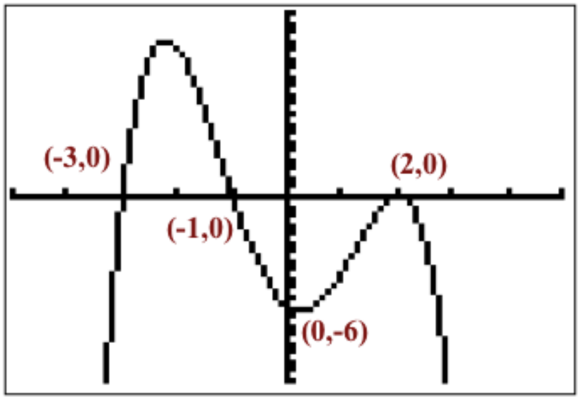
c) x-intercepts and the factors

d) the intervals where the function is positive or negative



EXAMPLE 1c - Analyse Graphs of Polynomial Functions

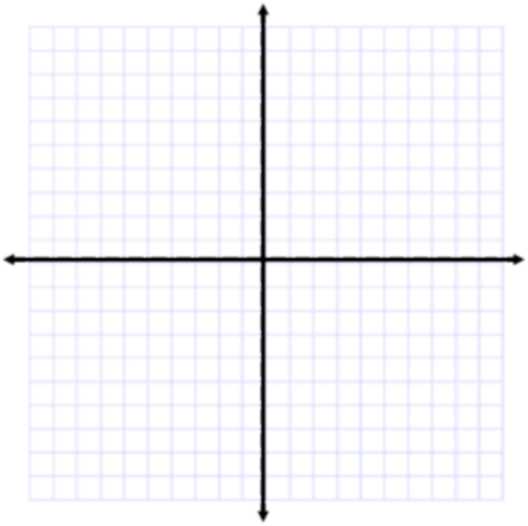
For the graph of a polynomial function, determine the equation.



EXAMPLE 2 - Analyse Equations to Sketch Graphs of Polynomial Functions

Sketch the graph of each polynomial function.

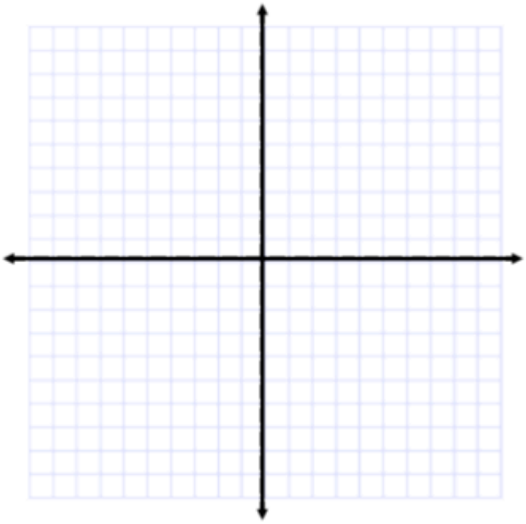
a) $y = (x-1)(x+2)(x+3)$



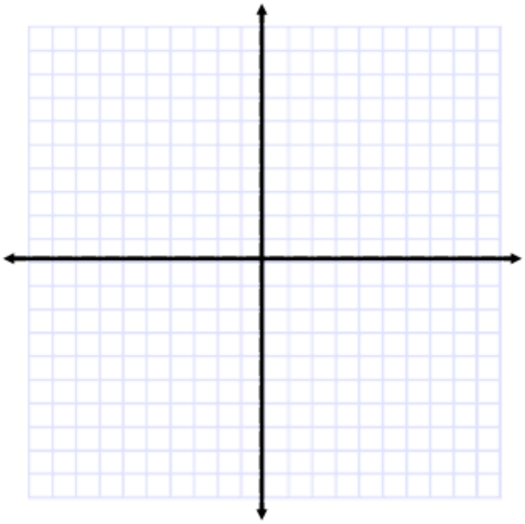
EXAMPLE 2b - Analyse Equations to Sketch Graphs of Polynomial Functions

Sketch the graph of each polynomial function.

b) $f(x) = -2x^3 + 6x - 4$



TRY: *Sketch the graph of the polynomial* $g(x) = -x^3 + 13x + 12$.

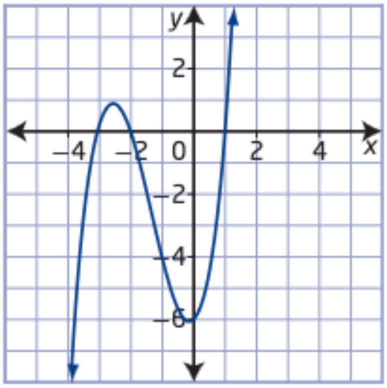


Complete Assignment Questions #3, #4 - #7

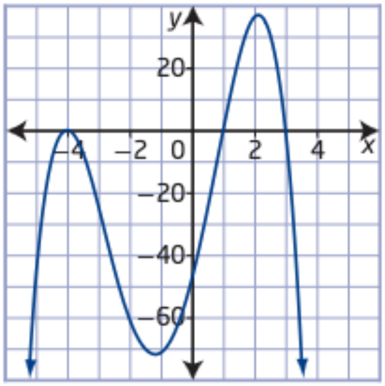
Assignment

3. Use the graph of the given function to write the corresponding polynomial possible equation. State the roots of the equation. The roots are all integral values.

a)



c)



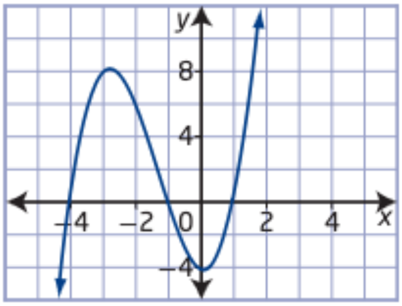
4. For each graph,

i) state the *x*-intercepts

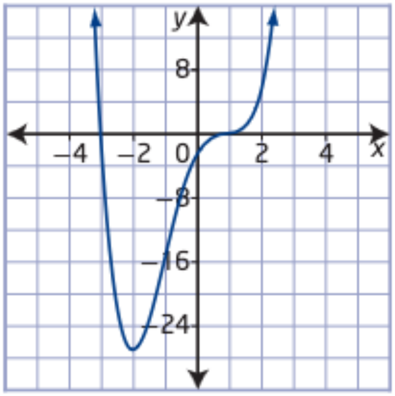
ii) state the intervals where the function is positive and the intervals where it is negative

iii) explain whether the graph might represent a polynomial that has zero(s) of multiplicity 1, 2, or 3

a)



c)



i)	
ii) pos neg	
iii)	

7. For each function, determine

i) the x-intercepts of the graph

ii) the degree and end behaviour of the graph

iii) the zeros and their multiplicity

iv) the y-intercept of the graph

v) the intervals where the function is positive and the intervals where it is negative

a) $y = x^3 - 4x^2 - 45x$

b) $f(x) = x^4 - 81x^2$

c) $h(x) = x^3 + 3x^2 - x - 3$

a)	b)	c)
i)		
ii)		
iii)		
iv)		
v) pos. neg.		

Answer Key

3. a) $(x + 3)(x + 2)(x - 1) = 0$, roots are -3, -2 and 1

c) $-(x + 4)^2(x - 1)(x - 3) = 0$, roots are -4, 1 and 3

4. a)
- **i)** -4, -1, and 1
- **ii)** positive for $-4 < x < -1$ and $x > 1$, negative for $x < -4$ and $-1 < x < 1$
- **iii)** all three zeros are of multiplicity 1, the sign of the function changes

c)
- **i)** -3 and 1
- **ii)** positive for $x < -3$ and $x > 1$, negative for $-3 < x < 1$
- **iii)** -3 (multiplicity 1) and 1 (multiplicity 3), at both the function changes sign but is flatter at $x = 1$

7. a)
- **i)** -5, 0, and 9
- **ii)** degree 3 from quadrant III to I
- **iii)** -5, 0, and 9 each of multiplicity 1
- **iv)** 0
- **v)** positive for $-5 < x < 0$ and $x > 9$, negative for $x < -5$ and $0 < x < 9$

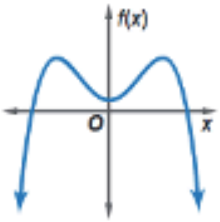
b)
- **i)** -9, 0 and 9
- **ii)** degree 4 opening upwards
- **iii)** 0 (multiplicity 2), -9 and 9 each of multiplicity 1
- **iv)** 0
- **v)** positive for $x < -9$ and $x > 9$, negative for $-9 < x < 9$, $x \neq 0$

c)
- **i)** -3, -1, and 1
- **ii)** degree 3 from quadrant III to I
- **iii)** -3, -1, and 1 each of multiplicity 1
- **iv)** -3
- **v)** positive for $-3 < x < -1$ and $x > 1$, negative for $x < -3$ and $-1 < x < 1$

LG 3 Worksheet (Polynomial Functions)

1. For each graph:
   i) describe the end behavior,
   ii) determine whether it represents an odd-degree or even-degree polynomial function, and
   iii) state the number of real zeros.

a.



b.

2. State the degree, the leading coefficient and the constant of each polynomial.

   a) $7 + 3x^2 - 5x^3 + 6x^2 - 2x$

   b) $(a + 1)(a^2 - 4)$

3. Use the Remainder Theorem, to find the remainder for:

   a) $4x^3 - 3x^2 + 2x - 4$ *is divided by* $x + 1$

   b) $y^3 + y^2 - 2y + 1$ *is divided by* $y - 3$

4. Given a polynomial and one of its factors, find the remaining factors using synthetic division.

   a) $2x^3 - 5x^2 - 28x + 15\,;\;\; x - 5$

   b) $x^4 - 3x^2 - 4\,;\;\; x + 2$

5. Find the value of *k* so that each remainder is 3.

   a) $(x^2 + kx - 17) \div (x - 2)$

   b) $(x^2 + 5x + 7) \div (x - k)$

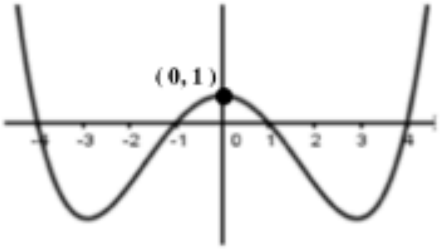
6. Factor completely each.

   a) $x^3 + 6x^2 - x - 30$

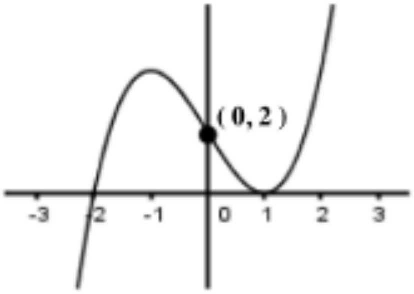
   b) $4x^3 + 8x^2 - x - 2$

Write an equation for graph below:

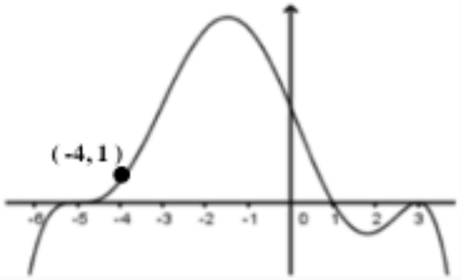
7.



8.

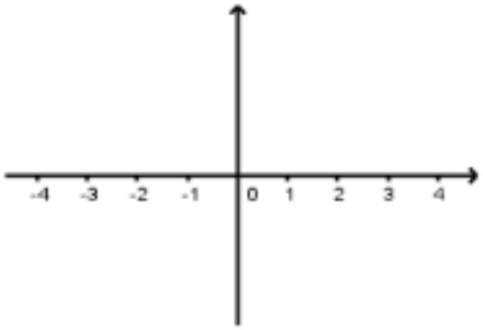


9.

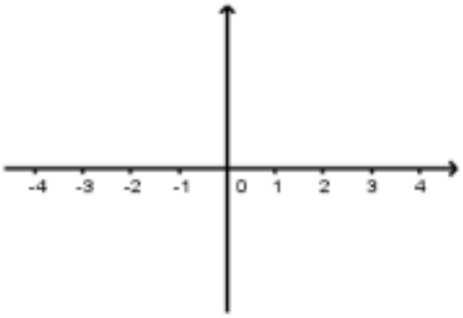


Sketch the graph of each polynomial using the information provided.

10. A polynomial with a negative leading coefficient and zeros of x = -2 (multiplicity 2) and x = 1.



11. A polynomial with a positive leading coefficient and zeros of x = -2 (multiplicity 3), x = 0, and x = 3 (multiplicity 2).



Answer Key

1. a) i) down Quad 3,4
ii) even
iii) 2

b) i) up Q1, down Q3
ii) odd
iii) 1

2. a) 3rd degree
-5
7

b) a) 3rd degree
1
-4

3. a) -13

b) 31

4. a) $(x+3)(2x-1)$ b) $(x-2)(x^2+1)$

5. a) $k=8$ b) $k=-4,-1$

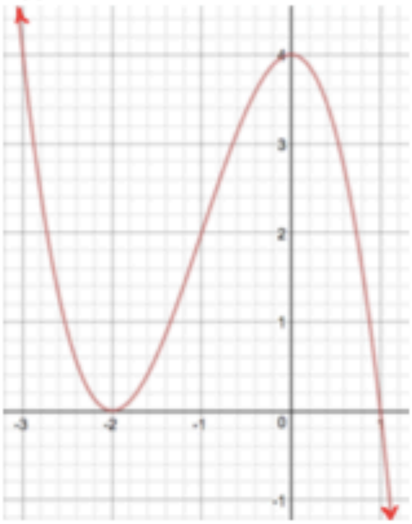
6. a) $(x-2)(x+3)(x+5)$
b) $(x+2)(2x+1)(2x-1)$

7. $y=\frac{1}{16}(x-1)(x+1)(x-4)(x+4)$

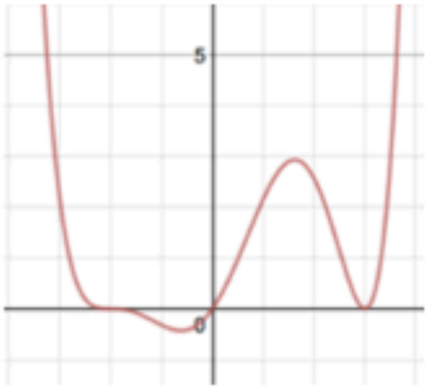
8. $y=(x+2)(x-1)^2$

9. $y=\frac{-1}{245}(x-1)(x-3)^2(x+5)^3$

10.



11.



LG 3 QUIZ (Polynomials)

1. **For the function $y = (x-1)(x+4)(x-3)$ determine the interval(s) where the function is negative.**

---

2. **Find the remainder when the polynomial $x^3 - 4x^2 - 2x - 1$ is divided by $x - 3$.**

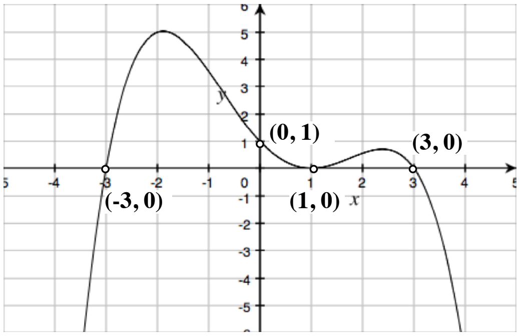
---

3. **The zeros of a cubic function are 1 and -3 (*multiplicity 2*). Determine the equation of the function that has these zeros and passes through the point (-2, 9)**

4. **Determine the value of k so that** $x-2$ **is a factor of** kx^3+x^2-8x-4.

---

5. **Use the graph of the given polynomial function to write its equation.**



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6. **For what value of *c* will the polynomial** $P(x) = 3x^3 + cx^2 - 9x + 2$ **have the same remainder when it is divided by** $(x+2)$ and $(x+1)$?

---

7. Factor $6x^3+x^2-21x-10$ **fully. Use synthetic or long division.**

*****SEE YOUR TEACHER FOR MARKING KEY*****

PRE-CALCULUS 12

Seminar Notes

Learning Guides 4 & 5

EXPONENTIAL & LOGARITHMIC FUNCTIONS

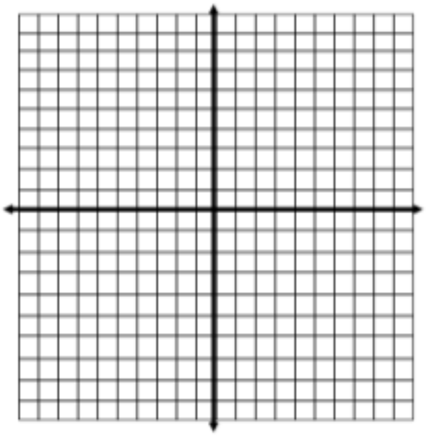
Like many types of functions, the exponential function has an inverse. This inverse is called the logarithmic function. Knowing about logarithmic functions is very useful for solving where *x* is in the exponent.

Topic 1 Characteristics of Exponential Functions

EXAMPLE 1 - Analyse the Graph of an Exponential Function

Complete the table and sketch the graph described by $y = 2^x$

x	y
–3	
–2	
–1	
0	
1	
2	
3	

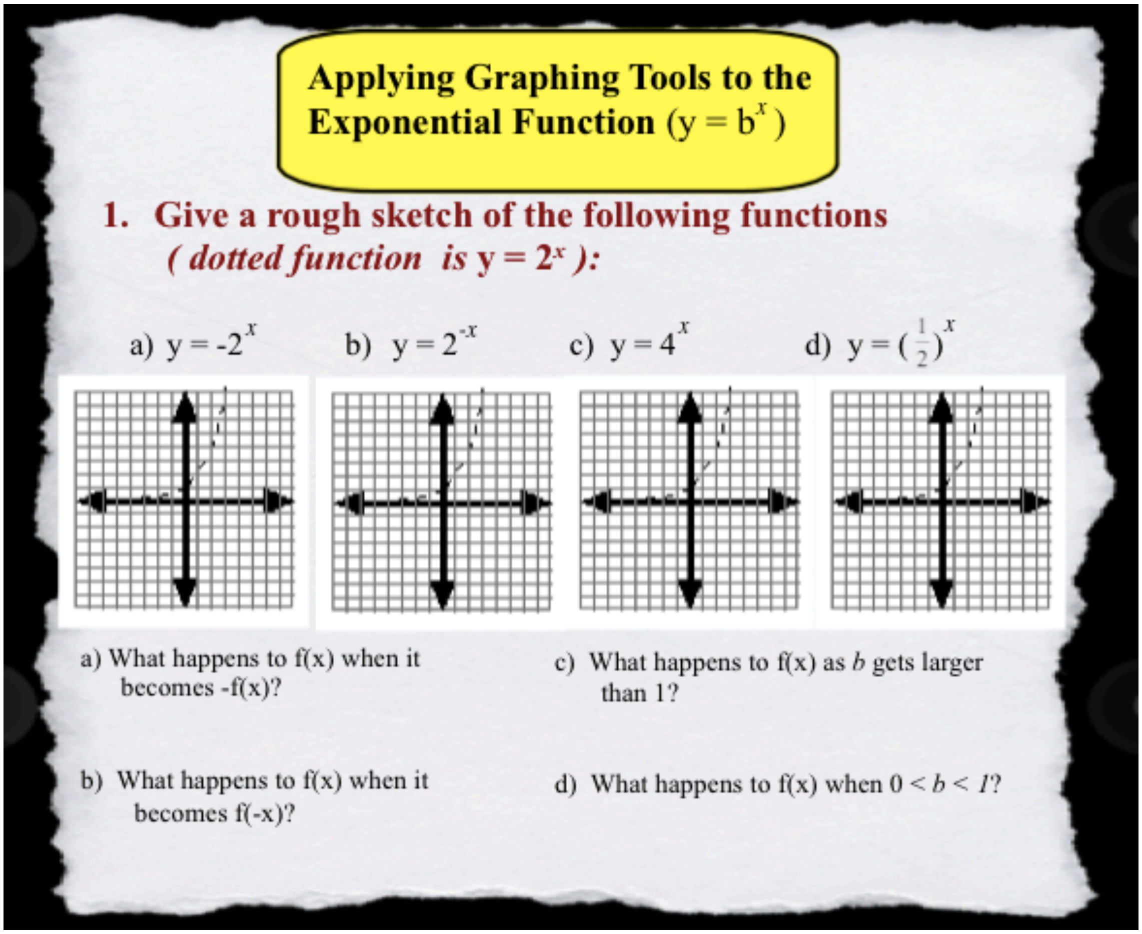


What is the x intercept? (let y =0)

What is the y-intercept? (let x = 0)

Domain: **Range:**

Asymptote:



Applying Graphing Tools to the Exponential Function ($y = b^x$)

1. Give a rough sketch of the following functions
(dotted function is $y = 2^x$):

a) $y = -2^x$ b) $y = 2^{-x}$ c) $y = 4^x$ d) $y = (\frac{1}{2})^x$

a) What happens to f(x) when it becomes -f(x)?

b) What happens to f(x) when it becomes f(-x)?

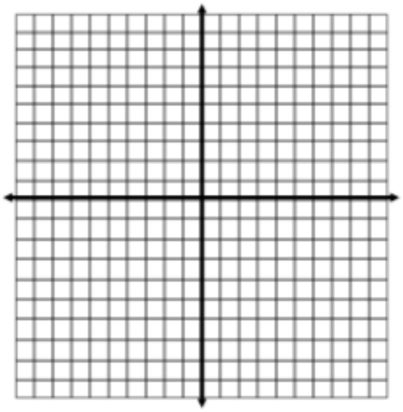
c) What happens to f(x) as b gets larger than 1?

d) What happens to f(x) when $0 < b < 1$?

TRY: Graph the following functions on the grid provided. Determine the equations of the asymptotes, find the values of the x- and y-intercepts and the domain and range.

$y = 3^x$ $y = 3^{x+1} - 2$

Hint: *Use your translation tools.*



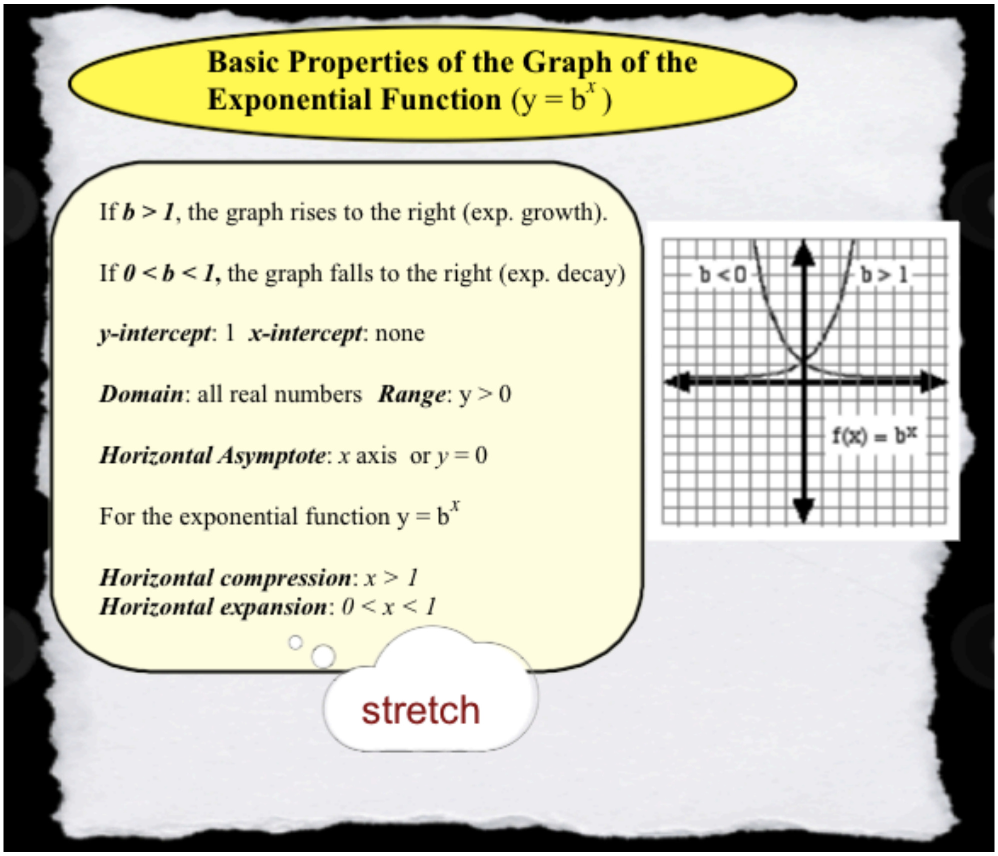
For $y = 3^{x+1} - 2$ determine:

Asymptote:

Domain and range

y-intercept (let $x = 0$)

x-intercept (let $y = 0$)



Basic Properties of the Graph of the Exponential Function ($y = b^x$)

If ***b* > *1***, the graph rises to the right (exp. growth).

If ***0* < *b* < *1***, the graph falls to the right (exp. decay)

y-intercept: 1 ***x-intercept***: none

Domain: all real numbers ***Range***: $y > 0$

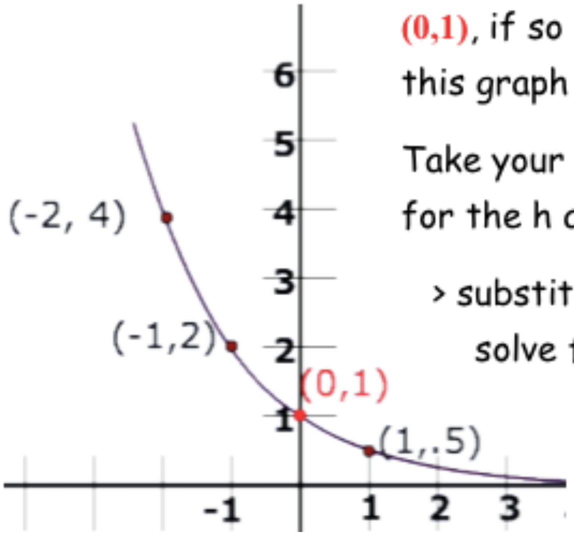
Horizontal Asymptote: x axis or $y = 0$

For the exponential function $y = b^x$

Horizontal compression: $x > 1$
Horizontal expansion: $0 < x < 1$

stretch

EXAMPLE 2 - Writing Exponential Function Given Its Graph

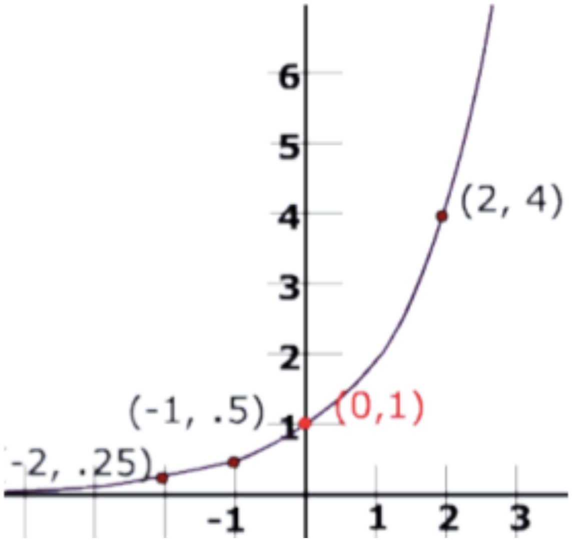


Solution: First see if the graph goes through (0,1), if so there is no translation happening. In this graph it does go through (0,1).

Take your base formula $\mathbf{y} = \mathbf{b}^{(x-h)} + \mathbf{k}$; no need for the h and k because no translation

> substitute x and y values into $\mathbf{y} = \mathbf{b}^x$ and solve for b

Try : Write an Exponential Function for the given Graph.



Topic 2 Transformations of Exponential Functions

$$f(x) = a(c)^{b(x-h)} + k$$

Parameter	Transformation
a	vertical stretch
b	horizontal stretch
k	vertical translation up or down
h	horizontal translation right or left

TRY: **Describe the roles of the parameters for the base function $f(x) = 2^x$. Then complete the table showing the transformations.**

i. $f(x) = 2^x + 3$

ii. $f(x) = 2^{x-1} + 5$

iii. $f(x) = \frac{1}{4}(2)^{-3x} - 2$

x	y
x	y
x	y

Apply Transformations to Sketch a Graph

EXAMPLE 1 - a) Sketch the base function $y = 3^x$ and then $y = -\frac{1}{2}(3)^{\frac{1}{5}x} - 5$ the transformed function.

b) State the domain and range of both functions.

Solution: **a)** ☛ First get base points from base function

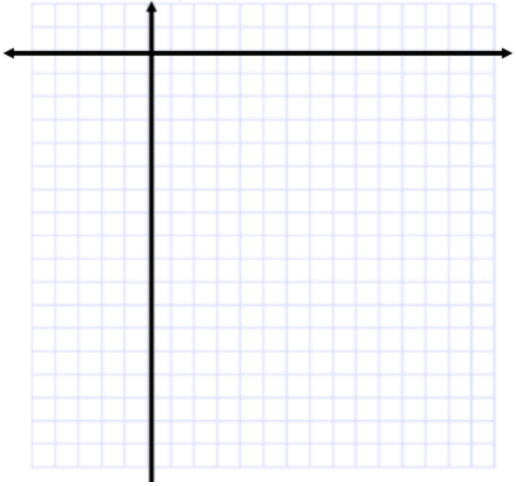
☛ Second, add two columns to represent the transformation of x and y

☛ Now plot the points of each function.

$y = 3^x$ $y = -\frac{1}{2}(3)^{\frac{1}{5}x} - 5$

touch ←

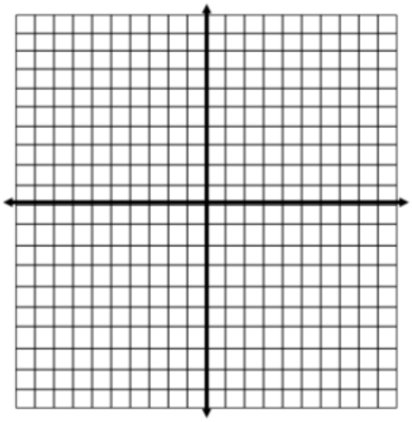
x	y	$5x$	$-\frac{1}{2}y-5$	
-1	1/3	-5	-31/6	-5.17
0	1	0	-11/2	-5.5
1	3	5	-13/2	-6.5
2	9	10	-19/2	-9.5
3	27	15	-37/2	-18.5



b) State the domain and range of both functions.

TRY: a) Transform the graph of y = 4^x to sketch the graph of y = $4^{-2(x+5)} - 3$.

b) State the domain and range of both.



Complete Assignment Questions #3 - #6

Assignment

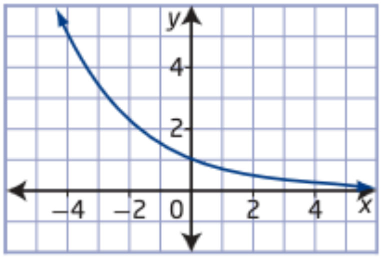
3. Match each exponential function to its corresponding graph.

a) $y = 5^x$

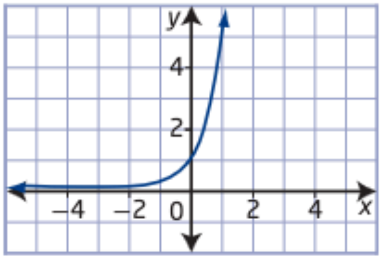
b) $y = \left(\frac{1}{4}\right)^x$

c) $y = \left(\frac{2}{3}\right)^x$

A



B



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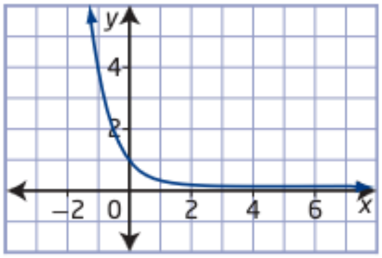
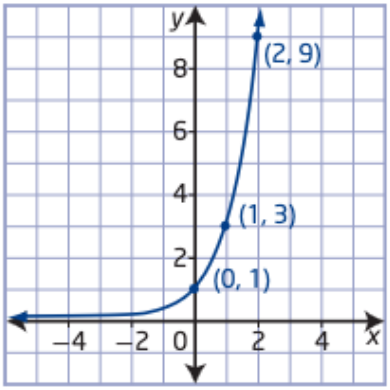
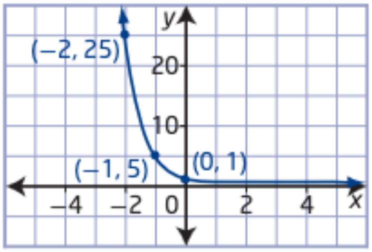


4. Write the function equation for each graph of an exponential function.

a)

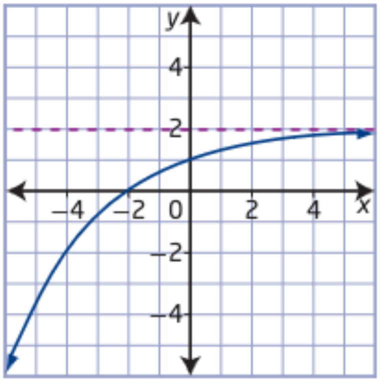


b)



5. Without using technology, match each graph with the corresponding function. Justify your choice.

a)



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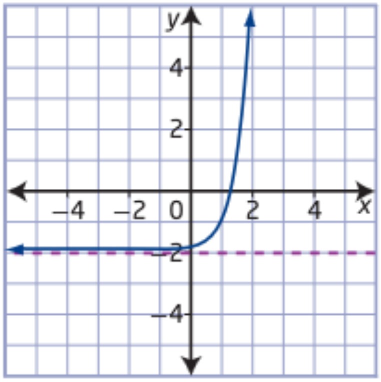

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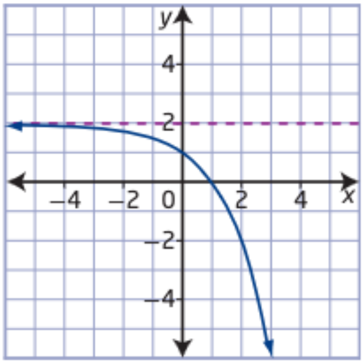

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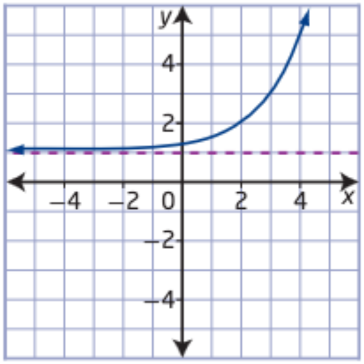


A $y = 3^{2(x-1)} - 2$

B $y = 2^{x-2} + 1$

C $y = -\left(\frac{1}{2}\right)^{\frac{1}{2}x} + 2$

D $y = -\frac{1}{2}(4)^{\frac{1}{2}(x+1)} + 2$

6. For each function,

i) state the parameters a, b, h, and k

ii) describe the transformation that corresponds to each parameter

iii) sketch the graph of the function

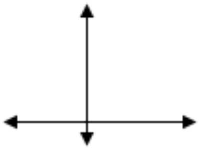
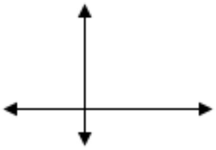
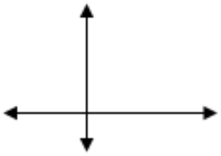
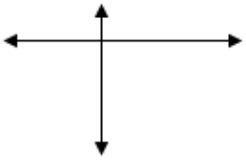
iv) identify the domain, range, equation of the horizontal asymptote, and any intercepts

a) $y = 2(3)^x + 4$

b) $m(r) = -(2)^{r-3} + 2$

c) $y = \frac{1}{3}(4)^{x+1} + 1$

d) $n(s) = -\frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{4}s} - 3$

a) i) ii)  iv)	b) i) ii)  iv)
c) i) ii)  iv)	d) i) ii)  iv)

Answer Key

3. a) B **b)** C **c)** A

4. a) $f(x) = 3^x$ **b)** $f(x) = \left(\frac{1}{5}\right)^x$

5. a) C: reflection in the x-axis, $a < 0$ and $0 < c < 1$, and vertical translation of 2 units up, $k = 2$

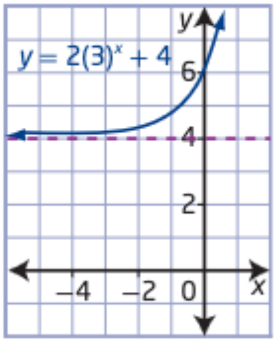
b) A: horizontal translation of 1 unit right, $h = 1$, and vertical translation of 2 units down, $k = -2$

c) D: reflection in the x-axis, $a < 0$ and $c > 1$, and vertical translation of 2 units up, $k = 2$

d) B: horizontal translation of 2 units right, $h = 2$, and vertical translation of 1 unit up, $k = 1$

6. a) i), ii) $a = 2$: vertical stretch by a factor of 2; $b = 1$: no horizontal stretch; $h = 0$: no horizontal translation; $k = 4$: vertical translation of 4 units up

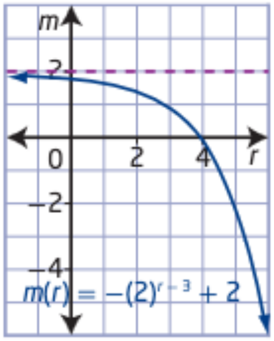
iii)



iv) domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y > 4, y \in \mathrm{R}\}$, horizontal asymptote $y = 4$, y-intercept 6

b) i), ii) $a = -1$: reflection in the x-axis; $b = 1$: no horizontal stretch; $h = 3$: horizontal translation of 3 units right; $k = 2$: vertical translation of 2 units up

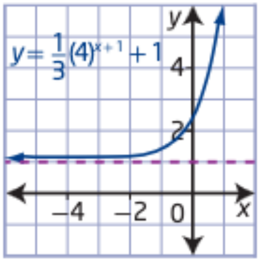
iii)



iv) domain $\{r \mid r \in \mathrm{R}\}$, range $\{m \mid m < 2, m \in \mathrm{R}\}$, horizontal asymptote $m = 2$, m-intercept $\frac{15}{8}$, r-intercept 4

c) i), ii) $a = \frac{1}{3}$: vertical stretch by a factor of $\frac{1}{3}$; $b = 1$: no horizontal stretch; $h = -1$: horizontal translation of 1 unit left; $k = 1$: vertical translation of 1 unit up

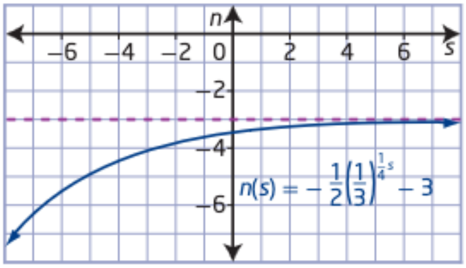
iii)



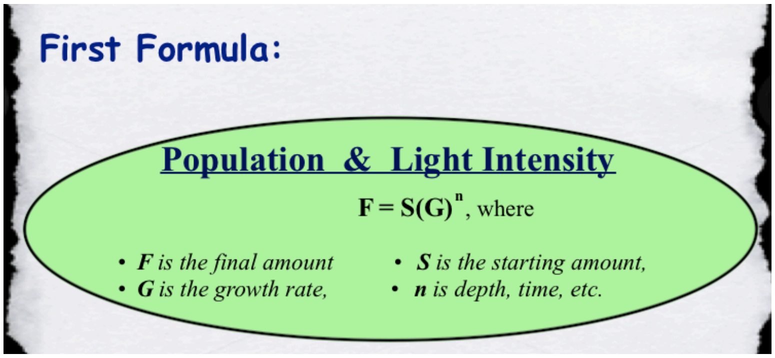
iv) domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y > 1, y \in \mathrm{R}\}$, horizontal asymptote $y = 1$, y-intercept $\frac{7}{3}$

d) i), ii) $a = -\frac{1}{2}$: vertical stretch by a factor of $\frac{1}{2}$ and a reflection in the x-axis; $b = \frac{1}{4}$: horizontal stretch by a factor of 4; $h = 0$: no horizontal translation; $k = -3$: vertical translation of 3 units down

iii)



iv) domain $\{s \mid s \in \mathrm{R}\}$, range $\{n \mid n < -3, n \in \mathrm{R}\}$, horizontal asymptote $n = -3$, n-intercept $-\frac{7}{2}$



First Formula:

Population & Light Intensity

$F = S(G)^n$, where

- ***F*** *is the final amount*
- ***G*** *is the growth rate,*
- ***S*** *is the starting amount,*
- ***n*** *is depth, time, etc.*

Example 1. The population, P in millions, of B.C. can be modelled by the equation:

$$P = 2.76(1.022)^n$$

➢ where n is the number of years since 1981.

a) Use a graphing calculator to sketch the function.

[0 , 50] [0 , 10]
xmin xmax ymin ymax

b) Trace to determine the population projected for 2021.

c) Estimate when the population might reach 5 million?

d) How long until the population doubles?

TRY: **When you consume caffeine, the percent, P, left in your body can be modelled as a function of the elapsed time, t hours, by the equation:**

$$\mathbf{P = 100(0.87)^t}$$

a) Use a graphing calculator to determine the percent of caffeine in your body after 5 hours.

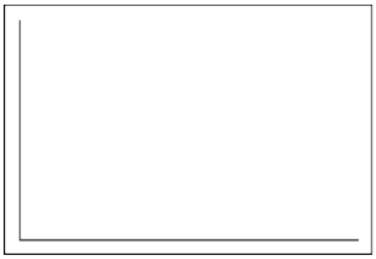
[0 , 10] [0 , 100]
xmin xmax ymin ymax

b) Estimate the number of hours that it will take until only 30% of the caffeine remains.

Example: **For every meter, n, that you descend into water, 5% of light is blocked. Use the formula to create an exponential function that gives the percent, F of light remaining for n meters.**

Note: Starting value is 100% because none of the light at the top has been blocked.

a) Write this equation and sketch the graph.



[0 , 50] [0 , 100]
xmin xmax ymin ymax

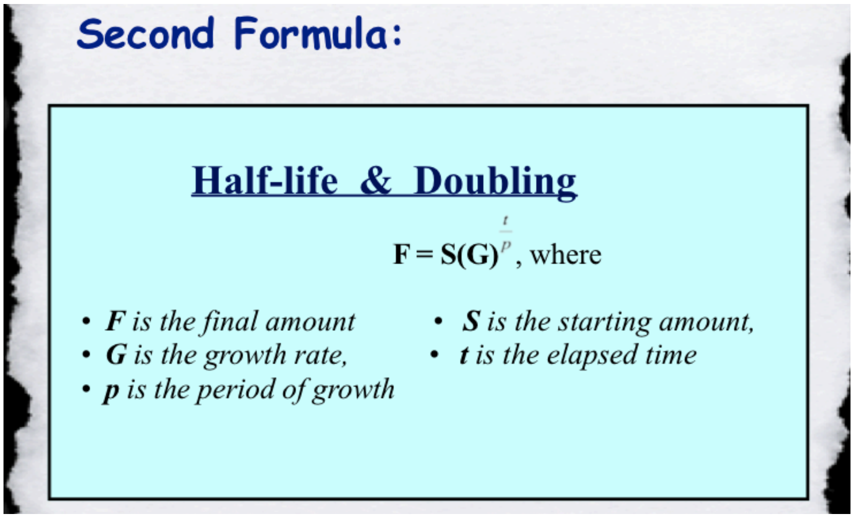
b) What percent of light remains at 6.5 meters?

c) At what depth below the surface is 50% of the light still remaining?

d) If 7% of the light were blocked for every meter you descend, rewrite the equation and determine how much light would remain at 5 meters below the surface.

TRY: A diver descends down to the bottom of a lake losing 3% of light per meter. Write an equation modelling a dive.

a) Find the percent of light at 20m. b) At what depth is there 40% of light?



Second Formula:

Half-life & Doubling

$F = S(G)^{\frac{t}{p}}$, where

- ***F*** *is the final amount*
- ***G*** *is the growth rate,*
- ***p*** *is the period of growth*
- ***S*** *is the starting amount,*
- ***t*** *is the elapsed time*

EXAMPLE 1 -

Radioactive phosphorus (P-32) is a common isotope used in DNA studies. It has a half- life of 14.3 days.

a) Use the formula above to write the equation which shows the percent P of this radioactive substance after t days.

b) How much time has elapsed if $\frac{1}{8}$ of he sample remains.

TRY: **If there are 600 bacteria in a culture initially. After 20 min there are 9600 bacteria. What is the doubling period of this bacterium?**

Complete Assignment Questions #1 - #12

Assignment

1. A swarm of killer bees triples in 30 days, the initial swarm had 1000 bees.

a) Determine the correct expression to calculate the number of bees after *t* days..

b) Find the number bees in a year?

c) Find the time in days if the population reached 27000?

2. A culture has 240 bacteria. The number of bacteria doubles every 6 hours. How long would it take for the culture to reach 7680 bacteria?

3. The population of Victoria today is 86,000. If the population continues at the growth rate of 2.7% per annum, how long will it take to reach 100,000 people.

4. A 200 g sample of radioactive polonium-210 has a half-life of 138 days.

a) Determine an equation to model this situation of mass of polonium, in grams, that remains after *t* days.

b). Determine the mass that remains after 5 years?

c) How long does it take for this 200 g sample to decay to 25 g?

5. If a bacteria's population doubles every 10 minutes, what does a population of 500 grow to in 40 minutes?

6. If a city's population increases by 3% per year, find the population in 10 years if the current population is 120,000 people.

7. If the half-life of I-44 is 8 minutes, how much of a 100 g sample is left after 40 minutes?

8. If a population of 100 bacteria grows to 3200 bacteria in 40 minutes, what is the doubling period of this bacterium?

9. If the tripling period of an insect is 12 days, how long does it take a population of 300 insects to grow to 8100?

10. If a sample of U-275 decays from 320 mg to 20 mg in 5 hours, what is the half-life of U-275?

11. If the half-life of B-127 is 18 seconds, how long does it take a sample of 400 g to decay to 25 g?

12. If a country's current population of 3.5 million people is decreasing at a rate of 1.9% per year, find its population in 20 years.

Answer Key

1. a) $y = 1000(3)^{t/30}$
   b) 63,822,7136 bees
   c) 90 days

2. 30 days

3. 5.7 years

4. a) $y = 200\left(\frac{1}{2}\right)^{t/138}$
   b) 0.02 g
   c) 414 days

5. 8000

6. 161270

7. 3.125 g

8. 8 min

9. 36 days

10. 1.25 hrs or 1 hr 15 min

11. 72 sec

12. 2.38 million

Topic 3

Solving Exponential Equations

EXAMPLE 1 - Change the Base of Powers

$a)\ 2x^4 = 32$. $b)\ 2x^{\frac{4}{5}} = 32$

$c)\ 16^{x+1} = 32^{2x-1}$. $d)\ 9^x = \left(\frac{1}{27}\right)^{2x+1}$

$e)\ 3^{2x} + 3(3^x) - 4 = 0$ $f)\ 4^{x-1} - 4^x + 12 = 0$

TRY:

$a)\ 2^{4x-1} = 8^x$. $b)\ \left(\frac{1}{125}\right)^{2x+1} = 25^x$

Complete Assignment Questions #1 - #6

Assignment

Solve each of the following exponential equations algebraically.

1. $4^{x-3} = 8^{2x+1}$

2. $16^{3x-1} = 4^{2x+5}$

3. $25^{2x-1} = \left(\frac{1}{5}\right)^{2x+5}$

4. $32^{3x-1} = \left(\frac{1}{8}\right)^{2x-2}$

5. $\left(\frac{1}{27}\right)^{2x+1} = \left(\frac{1}{9}\right)^{4x-2}$

6. $\left(\frac{1}{16}\right)^{2x-1} = \left(\frac{1}{8}\right)^{5x+2}$

Answer Key

1. $\frac{-9}{4}$ **2.** $\frac{7}{4}$ **3.** $\frac{-1}{2}$ **4.** $\frac{11}{21}$ **5.** $\frac{7}{2}$ **6.** $\frac{-10}{7}$

LG 4 Worksheet B (Exponential Equations)

Solving equations where *x* is in the exponent.

1. $4^{2x} - 6(4)^x + 8 = 0$

2. $2(8)^{2x} - 8^x = 6$

3. $2(2)^{-2x} - 9(2)^{-x} + 4 = 0$

4. $2^{x+3} + 2^{x+4} = 96$

5. $3^x - 3^{x-1} = 162$

Answer Key

1. $x = 1,\ x = \frac{1}{2}$
2. $x = \frac{1}{3}$
3. $x = 1,\ x = -2$
4. $x = 2$
5. $x = 5$

LG 4 QUIZ (EXPONENTIAL FUNCTIONS)

1. **Write the equation of the function that results from the following set of transformations: is $f(x) = 4^x$ stretched (*EXPANDED*) vertically by a factor of 2, stretched (*COMPRESSED*) horizontally by a factor of $\frac{1}{3}$, reflected in the y-axis, and translated 7 units up and 6 units to the left.**

---

2. **Describe in words how the function $y = \left(\frac{1}{3}\right)^x$ can be transformed into the function:**

$y - 1 = -6\left(\frac{1}{3}\right)^{2x-8}$.

---

3. **Given the function $y = -4(2)^{3x-3} + 5$ find each of the following:**

a) equations of any asymptote(s)

b) domain and range

c) x-intercepts and y-intercepts if they exist

---

4. **The function $T = 160\left(\frac{1}{2}\right)^{\frac{t}{10}}$ can be used to determine the length of time t, in hours, that milk will remain fresh where T is the storage temperature in degrees Celsius. How long will milk keep fresh at 20 degrees Celsius?**

5. **Solve algebraically:** $25^{2x+1} = 125^{4-3x}$

---

6. **Solve algebraically:** $\left(\frac{1}{8}\right)^{4x-1} = 16^{2x+5}$

---

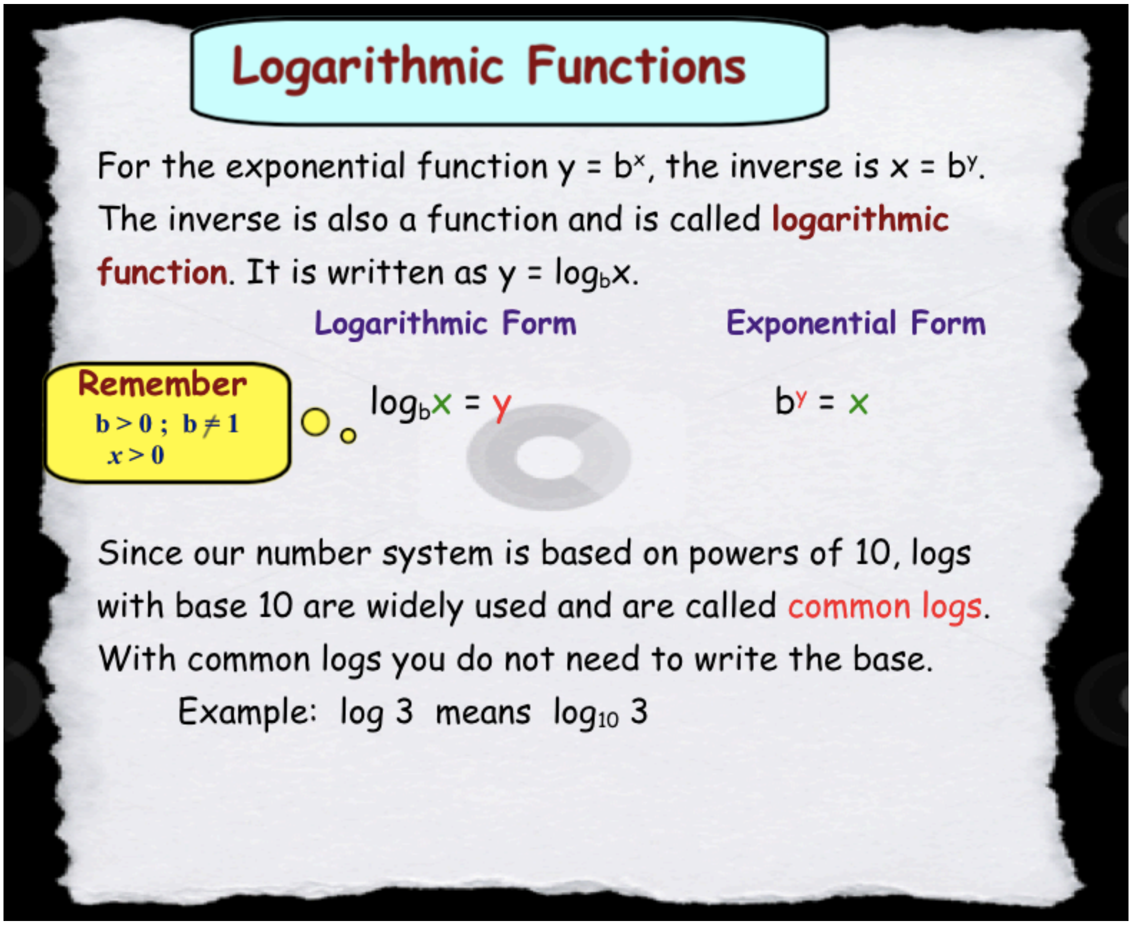
7. **A certain bacteria grew from a population of 100 to a population of 3200 in 8 hours. Find the doubling period of this bacterium.**

---

8. **If 12.5% of a sample of I-235 remains after 6 minutes, what is the half-life of I-235?**

*****SEE YOUR TEACHER FOR MARKING KEY*****

LEARNING GUIDE 5



Logarithmic Functions

For the exponential function $y = b^x$, the inverse is $x = b^y$. The inverse is also a function and is called **logarithmic function**. It is written as $y = \log_b x$.

Logarithmic Form	Exponential Form
$\log_b x = y$	$b^y = x$

Remember

$b > 0$; $b \neq 1$

$x > 0$

Since our number system is based on powers of 10, logs with base 10 are widely used and are called common logs. With common logs you do not need to write the base.

Example: $\log 3$ means $\log_{10} 3$

Topic 1 Understanding Logarithms

EXAMPLE 1 - Evaluating a Logarithm

Evaluate

Boot Up

a) $\log_5 25$ b) $\log 10$ c) $\log_8 1$

d) $\log_2 \sqrt{8}$ e) $\log_9 \sqrt[5]{81}$ f) $\log_3 9\sqrt{3}$

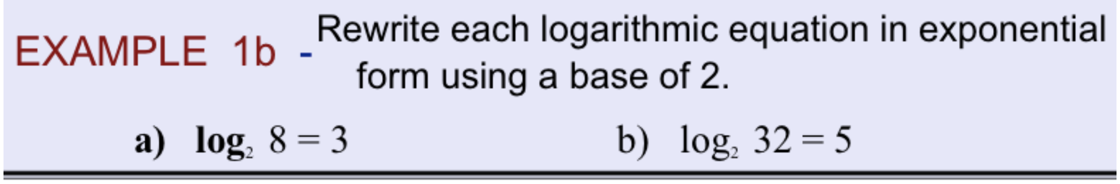
TRY: Evaluate:

a) $\log_2 64$ b) $\log 0.1$ c) $\log_5 125\sqrt{5}$

Not always can you get an exact answer.

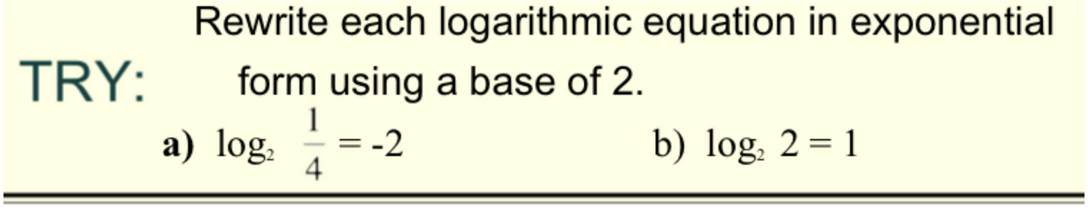
TRY: Use a calculator to determine each logarithm. Then write each equation using an exponent with base 10.

a) log 25 b) log 0.81 c) log (-2)



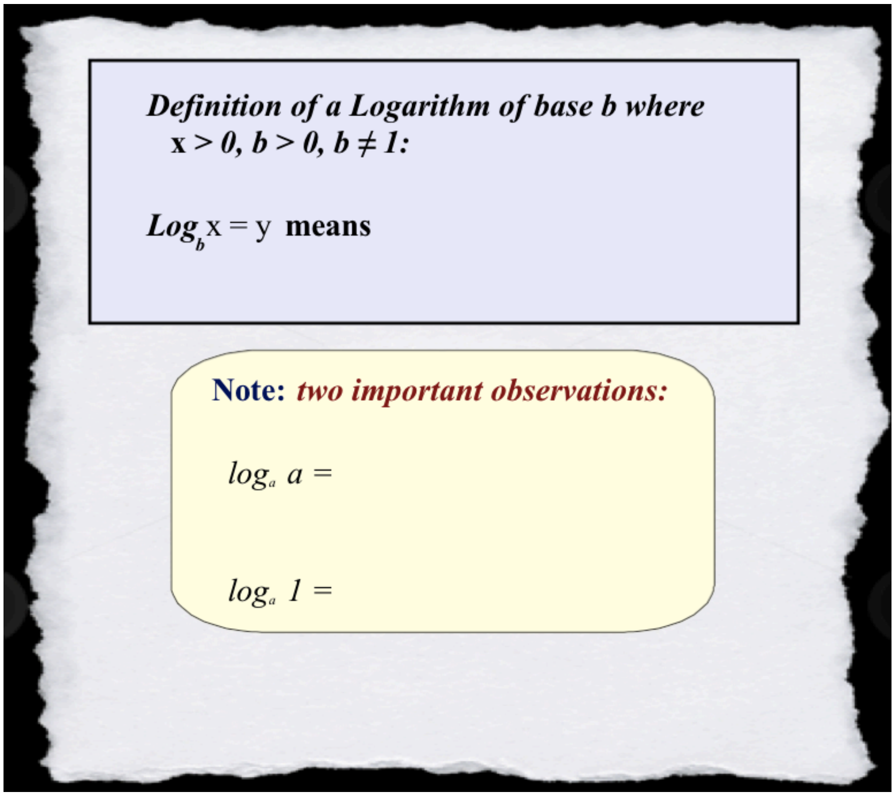
EXAMPLE 1b - Rewrite each logarithmic equation in exponential form using a base of 2.

a) $\log_2 8 = 3$ b) $\log_2 32 = 5$



TRY: Rewrite each logarithmic equation in exponential form using a base of 2.

a) $\log_2 \frac{1}{4} = -2$ b) $\log_2 2 = 1$



Definition of a Logarithm of base b where $x > 0, b > 0, b \neq 1$:

$Log_b x = y$ **means**

Note: ***two important observations:***

$log_a a =$

$log_a 1 =$

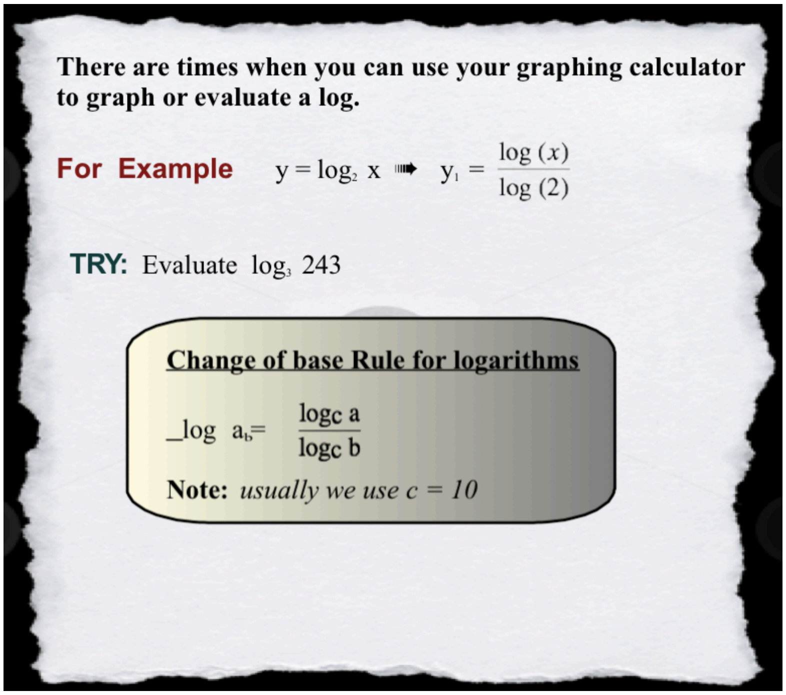
EXAMPLE 2 - Determine an Unknown in an Expression in Logarithmic Form

Determine the value of x.

a) $\log_5 x = -3$ b) $\log_x 36 = 2$ c) $\log_{64} x = \frac{2}{3}$

TRY: Determine the value of x.

a) $\log_4 x = -2$ b) $\log_{16} x = -\frac{1}{4}$ c) $\log_x 9 = \frac{2}{3}$



There are times when you can use your graphing calculator to graph or evaluate a log.

For Example $y = \log_2 x$ ➠ $y_1 = \frac{\log(x)}{\log(2)}$

TRY: Evaluate $\log_3 243$

Change of base Rule for logarithms

$\_\log\ a_b = \frac{\log c\ a}{\log c\ b}$

Note: *usually we use c = 10*

Complete Assignment Questions #3 - #9

Assignment

3. Express each of the following in logarithmic form.

a) $5^2 = 25$

b) $3^0 = 1$

c) $2^{-4} = \frac{1}{16}$

d) $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$

e) $b^d = e$

4. Express each of the following in exponential form.

a) $\log_3 9 = 2$

b) $\log_5 625 = 4$

c) $\log_4 \frac{1}{4} = -1$

d) $\log_a f = i$

e) $\log_{10} 0.001 = -3$

5. Evaluate.

a) $\log_4 64$ **b)** $\log_5 \sqrt{5}$ **c)** $\log_7 49$

d) $\log_{10} 0.001$ **e)** $\log_8 8^{-4}$ **f)** $\log_2 \sqrt{\frac{1}{512}}$

g) $\log_b 1$ **h)** $\log_c c$ **i)** $\log_x x^z$

6. Complete the following table:

Logarithmic Form	Exponential Form	Value of x
$\log_4 x = 2$		
	$7 = 49^x$	
$\log_x\left(\frac{1}{64}\right) = -3$		
	$x + 2 = 4^2$	
$\log_{32} x = \frac{1}{5}$		
	$\frac{1}{2} = 16^x$	

8. Solve for x.

a) $\log_x 125 = 3$ **b)** $\log_{125} 5 = x$ **c)** $\log_4 x = -8$

9. Find the inverse of the following equations. Answer in the form $y =$ ________ .

a) $y = 3^x$ **b)** $y = \log_4 x$ **c)** $y = 3x^2 + 2$

d) $y = \log_3 x$ **e)** $y = 20^x$ **f)** $x = 20^y$

Answer Key

3. **a)** $\log_5 25 = 2$ **b)** $\log_3 1 = 0$ **c)** $\log_2\left(\frac{1}{16}\right) = -4$ **d)** $\log_{\frac{1}{2}}\left(\frac{1}{16}\right) = 4$ **e)** $\log_b e = d$

4. **a)** $3^2 = 9$ **b)** $5^4 = 625$ **c)** $4^{-1} = \frac{1}{4}$ **d)** $a^i = f$ **e)** $10^{-3} = 0.001$

5. **a)** 3 **b)** $\frac{1}{2}$ **c)** 2 **d)** -3 **e)** -4 **f)** $\frac{-9}{2}$ **g)** 0 **h)** 1 **i)** z

6.

Logarithmic Form	Exponential Form	Value of x
$\log_4 x = 2$	$x = 4^2$	$x = 16$
$\log_{49} 7 = x$	$7 = 49^x$	$x = \frac{1}{2}$
$\log_x\left(\frac{1}{64}\right) = -3$	$\frac{1}{64} = x^{-3}$	$x = 4$
$\log_4(x+2) = 2$	$x + 2 = 4^2$	$x = 14$
$\log_{32} x = \frac{1}{5}$	$x = 32^{\frac{1}{5}}$	$x = 2$
$\log_{16}\left(\frac{1}{2}\right) = x$	$\frac{1}{2} = 16^x$	$x = -\frac{1}{4}$

8. **a)** 5 **b)** $\frac{1}{3}$ **c)** 0.0000153

9. **a)** $y = \log_3 x$ **b)** $y = 4^x$ **c)** $y = \pm\sqrt{\frac{x-2}{3}}$

d) $y = 3^x$ **e)** $y = \log_{20} x$ **f)** $y = 20^x$

EXAMPLE 3 -
Graph Exponential Function & Logarithmic Function

Comparing the Graphs of $y = 2^x$ and $y = \log_2 x$

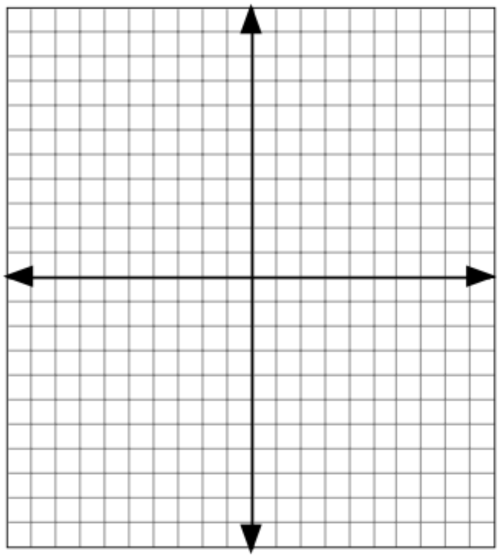
a) Construct the graph of the exponential function $y = 2^x$ and its inverse $y = \log_b x$, (or $x = 2^y$), $x, y \in R$, using the grid and tables of values provided below.

Graph of $y = 2^x$

x	-3	-2	-1	0	1	2	3	4
y								

Graph of $y = \log_2 x$, ($x = 2^y$)

x								
y	-3	-2	-1	0	1	2	3	4



b) Complete the table below

Function	Domain	Range	x-intercept	y-intercept	Asymptote
$y = 2^x$					
$y = \log_2 x$					

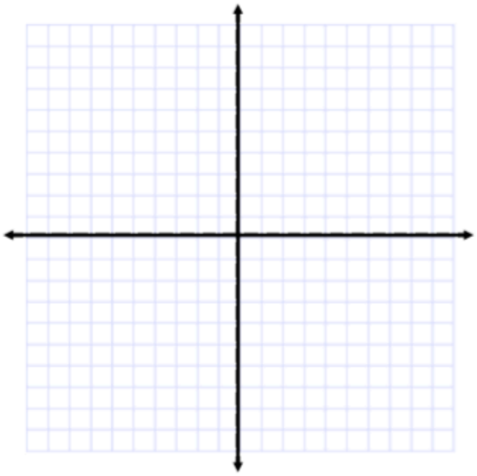
Topic 2 Transformations of Log Functions

EXAMPLE 1 - Translations of Log Functions

Graph on the grid provided. Determine the equations of the asymptotes and find the domain and range.

$y = \log_2 (x - 3) + 1$

Hint: *First find Asymptote,* then *Mapping*

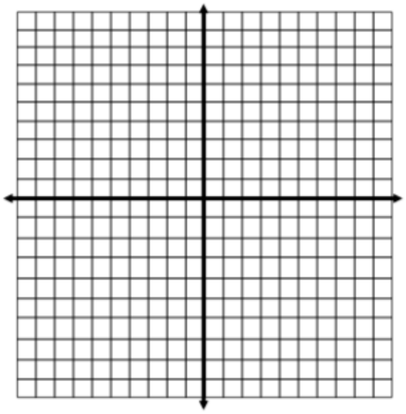


Domain: Range:

Asymptote:

TRY: Graph on the grid provided. Determine the equations of the asymptotes and find the domain and range.

$y = \log_2 (x + 2) - 4$



Domain: Range:

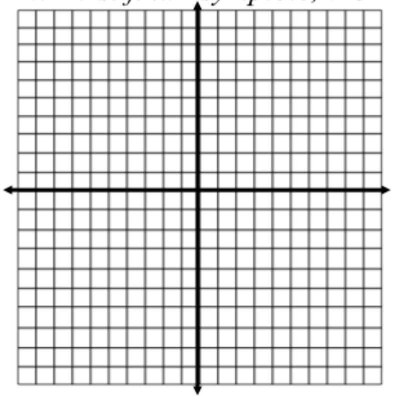
Asymptote:

EXAMPLE 2 - Reflections, Stretches, and Translations of a Log Function

Graph on the grid provided. Determine the equations of the asymptotes, find the domain and range and identify the y & x intercepts, if they exists.

$y = -\log_2 (2x + 6)$

Hint: *First find Asymptote,* then *Mapping*



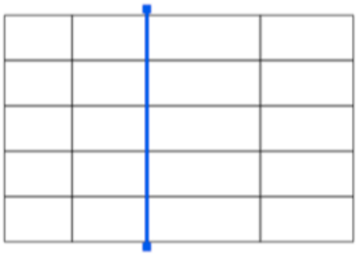
Domain: Range:

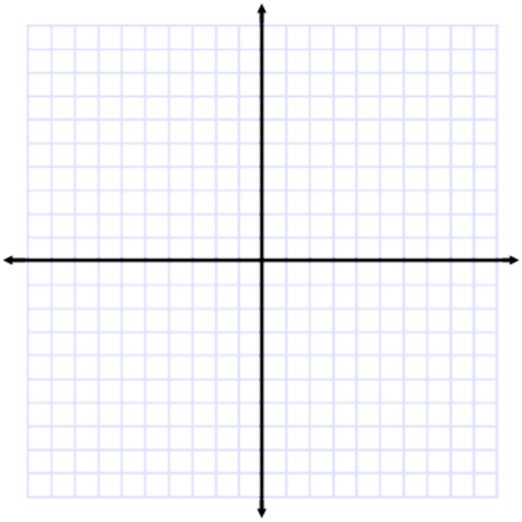
Asymptote:

x-int y-int

TRY: Graph on the grid provided. Determine the equations of the asymptotes, find the domain and range and identify the y & x intercepts, if they exists.

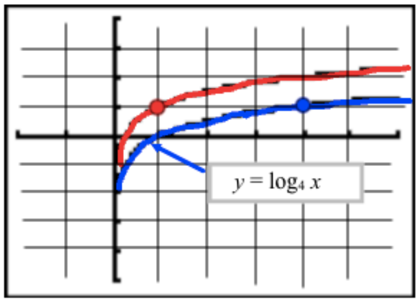
$y = 2\log_3 (-x + 1)$



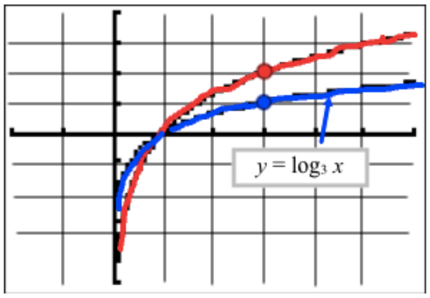


EXAMPLE 3 -
Determine the Equation of a Log Function Given Its Graph

a)



b)



The red graph is a stretch of the blue graph. Write the equation of each red graph.

Solution:

a) Use the standard form $y = \boldsymbol{a} \log_c \boldsymbol{b}x$

Notice the red graph has been horizontally stretched.

$y = \log_4 bx$ *use pt* $(1, 1)$

$1 = \log_4 b(1)$

$4^1 = b$ $\Rightarrow$ *answer* $\boxed{y = \log_4 4x}$

b) *Notice the red graph has been vertically stretched. Hint: x-intercept are the same.*

$y = a\log_3 x$ *use pt* $(3, 2)$

$2 = a\log_3 3$

$\frac{2}{a} = \frac{\cancel{a}}{\cancel{a}}\log_3 3$

$\frac{2}{a} = \log_3 3$

$\cancel{3}^{\frac{2}{a}} = \cancel{3}^1$

$\frac{2}{a} = 1$

$a = 2$ $\Rightarrow$ *ans.* $\boxed{y = 2\log_3 x}$

Complete Assignment Questions #1 - #6

Assignment

1. Describe how the graph of each logarithmic function can be obtained from the graph of $y = \log_5 x$.

a) $y = \log_5 (x - 1) + 6$

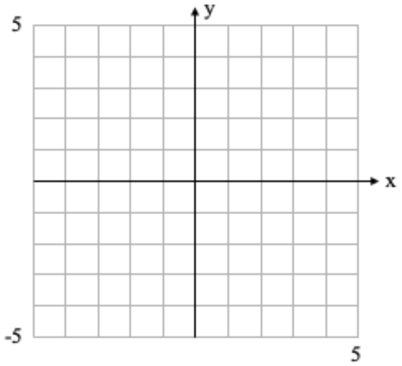
b) $y = -4 \log_5 3x$

c) $y = \frac{1}{2} \log_5 (-x) + 7$

2. a) Sketch the graph of $y = \log_3 x$, and then apply, in order, each of the following transformations.

- Stretch vertically by a factor of 2 about the x-axis.
- Translate 3 units to the left.

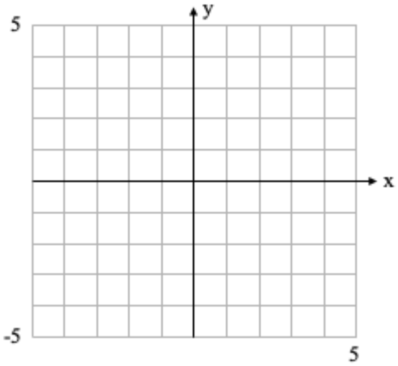
b) Write the equation of the final transformed image.



3. a) Sketch the graph of $y = \log_2 x$, and then apply, in order, each of the following transformations.

- Reflect in the y-axis.
- Translate vertically 5 units up.

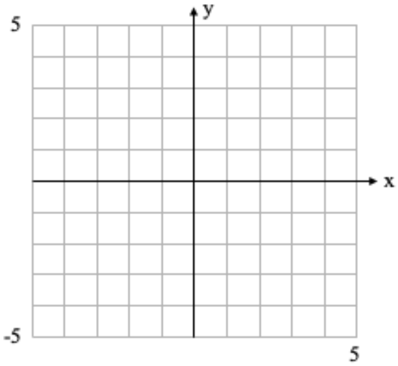
b) Write the equation of the final transformed image.



4. Sketch the graph of each function.

a) $y = \log_2 (x + 4) - 3$

b) $y = -\log_3 (x + 1) + 2$



5. Identify the following characteristics of the graph of each function.

i) the equation of the asymptote

ii) the domain and range

iii) the y-intercept, to one decimal place if necessary

iv) the x-intercept, to one decimal place if necessary

a) $y = -5 \log_3 (x + 3)$

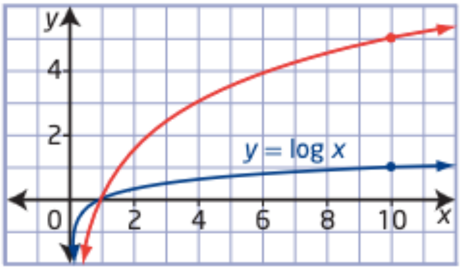
b) $y = \log_6 (4(x + 9))$

c) $y = \log_5 (x + 3) - 2$

	a)	**b)**	**c)**
i)			
ii)			
iii)			
iv)			

6. In each, the red graph is a stretch of the blue graph. Write the equation of each red graph.

a)



b)

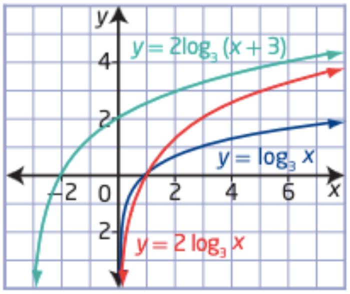
Answer Key

1. a) Translate 1 unit right and 6 units up.

b) Reflect in the x-axis, stretch vertically about the x-axis by a factor of 4, and stretch horizontally about the y-axis by a factor of $\frac{1}{3}$.

c) Reflect in the y-axis, stretch vertically about the x-axis by a factor of $\frac{1}{2}$, and translate 7 units up.

2. a)



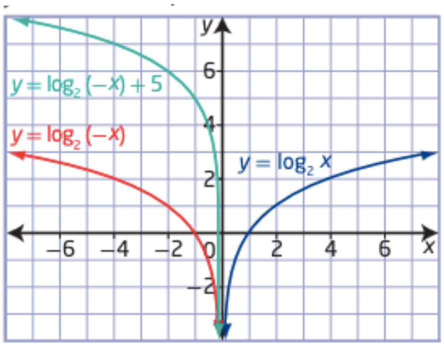
b) $y = 2 \log_3 (x + 3)$

5. a) **i)** vertical asymptote $x = -3$
ii) domain $\{x \mid x > -3, x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$
iii) y-intercept -5 **iv)** x-intercept -2

b) **i)** vertical asymptote $x = -9$
ii) domain $\{x \mid x > -9, x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$
iii) y-intercept 2 **iv)** x-intercept -8.75

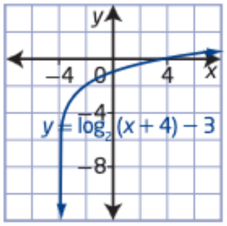
c) **i)** vertical asymptote $x = -3$
ii) domain $\{x \mid x > -3, x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$
iii) y-intercept -1.3 **iv)** x-intercept 22

3. a)

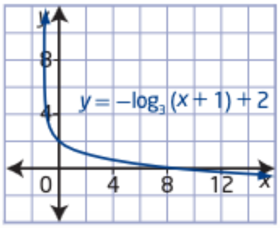


b) $y = \log_2 (-x) + 5$

4. a)



b)



6. a) $y = 5 \log x$ **b)** $y = \log_8 2x$

Topic 3 Laws of Logarithms

EXAMPLE 1 - Laws of Logarithms for Powers

Write an expressions that is equal to the following expressions.

$\log \pi^4$ ➠

$\log \sqrt{x}$ ➠

TRY:

$\log m^n$ ➠

$\log_2 8^{\frac{1}{2}}$ ➠

TRY: Use this rule to simplify the following expressions:

$\log x^5$ ➠

$\log \sqrt[5]{y}$ ➠

$\log_2 x^{\frac{3}{4}}$ ➠

Remember :
Roots are Powers
ex. $\sqrt[x]{b^y} = b^{\frac{y}{x}}$

$\sqrt[3]{7} = 7^{\frac{1}{3}}$

There are times the power law of logs are used in a slightly different way to simplify a problem.

EXAMPLE 1b - Simplify $\log_m m^4$

$= \log_m m^4$

$= 4\cancel{\log_m m}$

$\log_m m = 1$

$= 4$

TRY: Simplify the following.

Hint: Think Bases =

a) $\log_7 49^5$

b) $\log_8 2^x$

c) $\log_{\frac{1}{2}} 16$

Do worksheet "Conjugate Rule"

EXAMPLE 1c - Product & Quotient Laws of Logarithms

Write an expressions that is equal to the following expressions.

log (2.7 x 5) ➡

$\log_5 \frac{\pi}{2}$ ➡

TRY: Write an expressions that is equal to the following expressions.

$\log_3 (4 \times 7)$ ➡

$\log_2 \frac{3}{8}$ ➡

Law of logarithms for Product and Quotient

$\mathbf{log}_a\, xy =$

$\log_a \frac{x}{y} =$

where $a > 0, a \neq 1$

TRY:

a) Write log 6 as a sum of 2 logarithms.

b) Write log 6 as a difference of 2 logarithms.

Knowing yours Log Laws are important here!

EXAMPLE 2 - Write as a Single Logarithm

$\log 4 + \log 5$ ➠ $\log (4\times5)$ ➠ $\log 20$

$\log 32 - \log 8$ ➠ $\log (32\div8)$ ➠ $\log 4$

TRY: Write as a Single Logarithm

$\log a + \log b - \log c$ ➠

Write in terms of log a, log b, log c

$\log \dfrac{a}{b\sqrt{c}}$ ➠

TRY: Use the Laws of Logarithm to simplify and then evaluate each expression.

a) $\log_6 8 + \log_6 9 - \log_6 2$

b) $\log_3 9\sqrt{3}$

EXAMPLE 3 - Use the Laws of Logarithm to Simplify Expressions

Write each expression as a singles logarithm in simplest form. State the restriction on the variable.

$a)\ \log_7 x^2 + \log_7 x - \frac{5\log_7 x}{2}$ $\qquad$ $b)\ \log_5(2x-2) - \log_5(x^2+2x-3)$

Solution:

$$= \log_7 x^2 + \log_7 x - \overset{\curvearrowright}{\frac{5}{2}\log_7 x}$$

$$= \log_7 x^2 + \log_7 x - \log_7 x^{\frac{5}{2}}$$

$$= \log_7 \frac{(x^2)(x)}{x^{\frac{5}{2}}}$$

$$= \log_7 x^{2+1-\frac{5}{2}}$$

$$= \overset{\curvearrowleft}{\log_7 x^{\frac{1}{2}}}$$

$$= \frac{1}{2}\log_7 x, \quad x > 0$$

Restriction

Solution:

$$= \log_5 \frac{2x-2}{x^2+2x-3}$$

Factor

$$= \log_5 \frac{2\cancel{(x-1)}}{(x+3)\cancel{(x-1)}}$$

$$= \log_5 \frac{2}{x+3} \quad , \ x > 1$$

Restriction

$2x-2>0$	$x^2+2x-3>0$
$2x>2$	$(x+3)(x-1)>0$
$x>1$	$x<-3 \text{ or } x>1$
$\therefore\ x>1$	

TRY: **Write each expression as a single logarithm in simplest form. State the restrictions on the variable.**

$a)\ 4\log_3 x - \frac{1}{2}(\log_3 x + 5\log_3 x)$ $\qquad$ $b)\ \log_2(x^2-9) - \log_2(x^2-x-6)$

EXAMPLE 4 - Determine an Expression in Specific T

Knowing yours Log Laws are important here!

If $\log 5 = x$ and $\log 4 = y$

a) $\log 20$ ➠ $\log 4 \times \log 5$ ➠ $x + y$

b) log 100 ➠ $\log 25 \times \log 4$ ➠ $\overset{\curvearrowleft}{\log 5^{2}} \times \log 4$
➠ $2\log 5 + \log 4$
➠ $2x + y$

Complete Assignment Questions #1 - #6

Assignment

1. Use the laws of logarithms to evaluate each of the following.

a) $\log_2 4 + \log_2 8$

b) $\log_4\left(\frac{7}{2}\right) - \log_4 56$

c) $\log_6 9 + \log_6 8 - \log_6 2$

d) $\log 2 + \log 10 - \log \frac{1}{5}$

2. Use the laws of logarithms to evaluate each of the following.

a) $\log_2\left(\frac{4}{3}\right) + \log_2 768$

b) $\log 8 + \log 5 - \log \frac{2}{5}$

c) $\log 3 + \log 4 + \log \frac{1}{2} + \log \frac{1}{6}$

d) $\log_3 3 + \log_3 2 - \log_3 27 - \log_3 6$

e) $\log_3 9 - \log_3\left(\frac{1}{3}\right)$

f) $\log_2(2^8) + \log_2\left(\frac{1}{8}\right)^2$

3. Which of the following are true and which are false for logarithms to ***every base***?

a) $\log 2 + \log 3 = \log 5$ **b)** $\log 3 + \log 4 = \log 12$ **c)** $\log 8 = \log 4 + \log 2$

d) $\log 10 + \log 10 = \log 100$ **e)** $\log 2 \times \log 3 = \log 6$ **f)** $\dfrac{\log 8}{\log 2} = \log 4$

g) $\log 3^2 + \log 3^{-2} = 0$ **h)** $\log \dfrac{5}{3} = \dfrac{\log 5}{\log 3}$ **i)** $\log \dfrac{1}{8} = -\log 8$

4. Use the laws of logarithms to evaluate the following.

a) $\log_2 \sqrt[3]{16}$ **b)** $\log_3 27^{\frac{1}{2}}$

5. Write each expression as a single logarithm:

a) $\log x - 3 \log y - 2 \log z$ **b)** $\dfrac{1}{3} \log_a p + 3 \log_a q - 4 \log_a p$

6. Simplify the following:

a) $\log 2 + 2 \log 3 - \log 18$ **b)** $2\log_4 2 - 2\log_4 4 - \log_4 \dfrac{1}{4}$

7. **a)** Show that $\log 81 = 4 \log 3$.

b) Hence simplify: **(i)** $\log 81 - \log 27$ **(ii)** $\dfrac{\log 81}{\log 27}$

8. Which of the following are true and which are false for logarithms to ***every base***?

a) $\log 5^{-2} = -2 \log 5$ **b)** $\log 4 = \frac{2}{3} \log 8$ **c)** $\log 27 = \frac{3}{4} \log 81$

d) $\frac{1}{3} \log 11 = \log \frac{11}{3}$ **e)** $\log 5 = \frac{1}{2} \log 10$ **f)** $\dfrac{\log \sqrt{2}}{\log \sqrt{8}} = \dfrac{1}{3}$

g) $\log \frac{1}{5} - \log 5 = -\log 25$ **h)** $\log 2 - \log \sqrt{2} = \log \sqrt{2}$

9. Which is the greatest of $\frac{2}{3} \log 1, \frac{3}{4} \log 1, \frac{4}{3} \log 1$?

10. Simplify the following:

a) $\log x^4 - 3 \log x + \log \frac{1}{x}$ **b)** $\log x^{\frac{1}{2}} + \log y^{\frac{1}{2}} - \frac{1}{2} \log xy$

c) $\log_2\sqrt{6} - \frac{1}{2}\log_2 3$

d) $\frac{1}{2}\log_{10}10 + 3\log_{10}\sqrt{10}$

e) $\log_a a + \log_c a^5 - \log_c a$

f) $\frac{2}{3}\log_2 a - 5\log_2 b - \frac{1}{5}\log_2 c^3$

g) $\log_a y^{2x-3} + \log_a y^{5x-2} - \log_a y^{x-5} - 2\log_a y^{3x+1}$

11. Simplify.

a) $\log_5 5^7$ **b)** $10^{\log 6}$ **c)** $\ln e^4$ **d)** $\log_c c^t$ **e)** $e^{\ln 7}$

f) $(5^{\log_5 2})(5^{\log_5 3})$ **g)** $\dfrac{\left(\sqrt{2}^{\;\log_6 27}\right)\left(\sqrt{2}^{\;\log_6 16}\right)}{\sqrt{2}^{\;\log_6 12}}$

12. $\log x + \log(x+4)$ is equal to

A. $\log(2x+4)$

B. $\log(x^2+4x)$

C. $\log(x^2+4)$

D. $\log(x)\log(x+4)$

13. $\log(x^2-4) - \log(x-2)$ is equal to

A. $\log(x+2)$

B. $\log(x^2-x-2)$

C. $\log(x-2)$

D. $\dfrac{\log(x^2-4)}{\log(x-2)}$

14. The expression $3\log_x 4 + \log_x 8 - \frac{1}{4}\log_x 16$, where $x > 0$, is equal to

A. $\log_x 384$

B. $\frac{3}{4}\log_x 512$

C. $\log_x 256$

D. $\frac{1}{4}\log_x\left(\frac{1}{2}\right)$

Answer Key

1. **a)** 5 **b)** –2 **c)** 2 **d)** 2

2. **a)** 10 **b)** 2 **c)** 0 **d)** –3 **e)** 3 **f)** 2 **g)** 0

3. **a)** F **b)** T **c)** T **d)** T **e)** F **f)** F **g)** T **h)** F **i)** T

4. **a)** $\frac{4}{3}$ **b)** $\frac{3}{2}$ **5.** **a)** $\log\left(\frac{x}{y^3z^2}\right)$ **b)** $\log_a\frac{q^3}{p^{\frac{11}{3}}}$ **6.** **a)** 0 **b)** 0

7. **b)** **(i)** log 3 **(ii)** $\frac{4}{3}$ **8.** **a)** T **b)** T **c)** T **d)** F **e)** F **f)** T **g)** T **h)** T

9. none because each of these equal zero.

10. **a)** 0 **b)** 0 **c)** $\frac{1}{2}$ **d)** 2 **e)** $1 + \log_c a^4$ **f)** $\log_2\left(\frac{a^{\frac{2}{3}}}{b^5c^{\frac{3}{5}}}\right)$ **g)** $-2\log_a y$ or $\log_a\frac{1}{y^2}$

11. **a)** 7 **b)** 6 **c)** 4 **d)** t **e)** 7 **f)** 6 **g)** 2

12. B **13.** A **14.** C

EXAMPLE 5 - **Determine an Expression in Specific Terms**

If $\log_3 2 = k$, write an algebraic expression in terms of k for each of the following.

a) $\log_3 54$ *b*) $\log_3 72$ *c*) $\log_3 \frac{16}{81}$ *d*) $\log_3 \sqrt{27}$

EXAMPLE 5 - **Determine an Expression in Specific Terms**

$a)$ *If* $\log_6 5 = m$, write an algebraic expression in terms m for $\log_{\frac{1}{6}} 30$

$b)$ *If* $\log_2 x = 6$; *evaluate* $\log_8 32x^2$

TRY: If log 2 = m and log 3 = n, determine an expression in terms of m and n for each of the following:

$a)$ $\log 60$

$b)$ $\log \frac{4}{9}$

LG 5 Worksheet B (Base Change & Log Laws)

Write an exact expression for each of the following using base change then evaluate each to 4 decimal places.

1. $\log_3(8)$

2. $\log_5\left(\frac{1}{8}\right)$

3. $\log_{\frac{1}{2}}(24)$

Write each expression in terms of individual logarithms of x, y, & z.

4. $\log_7(x^2y^3z)$

5. $\log_7\left(\frac{x^2}{y^3\sqrt{z}}\right)$

6. $\log_7(x^2yz)^3$

Write each expression as a single logarithm in reduced form and state any restrictions

7. $\log(A) + 2\log(B) - \frac{1}{3}\log(C)$

8. $3\log(C) - 4\log(D) - \frac{1}{4}\log(E)$

9. $-\log_2(E) + \log_2(F) - 3\log_2(K)$

10. $\log_3(A) - 2\log_3(B) + 3\log_3(C) - \frac{1}{4}\log_3(D)$

11. $2\log(x) - 3\log(y) - 4\log(z)$

Simplify

12. $\log_4 4^x$

13. $5^{\log_5 y^2}$

14. $\log_a a^8$

15. $b^{\log_b 7}$

16. $x^{3\log_x 2}$

17. $y^{-2\log_y 3}$

Simplify.

18. $\log_a a^7$

19. $\log_{a^3} a$

20. $\log_{a^4} a^8$

21. $\log_{a^2} \frac{1}{a^6}$

22. $\log_{a^m} a^k$

23. $\log_4 8$

24. $\log_9 27$

25. $\log_9 8$

Evaluate the following

26. If $\log_5 x = 25$, evaluate $\log_5 \frac{x}{25}$

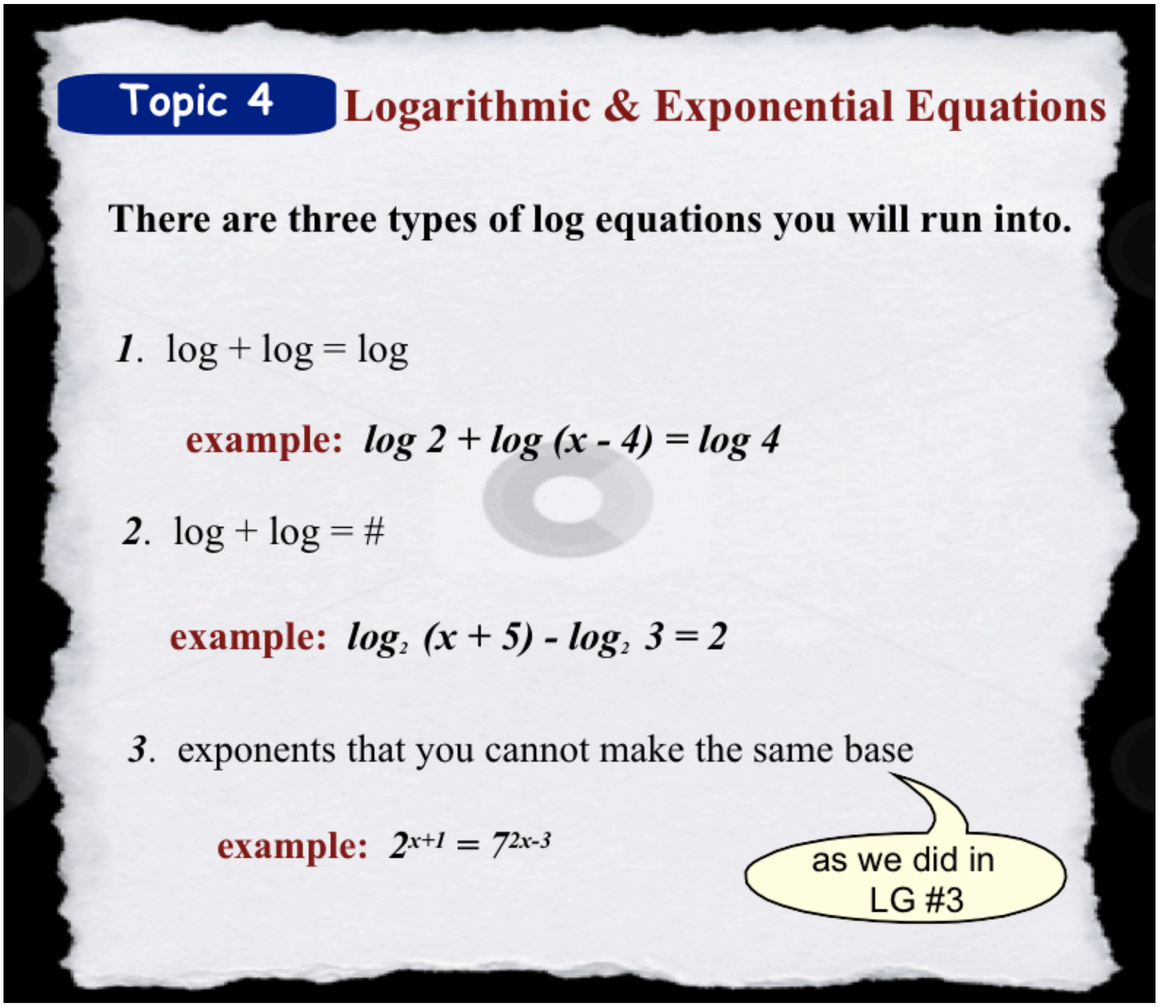
27. If $\log c = 3$, evaluate $\log c^2$

28. If $\log_4 x = a$, evaluate $\log_{16} x$

29. If $\log_n a = 5$ and $\log_n b = 3$, evaluate $\log_n ab^2$

Answer Key

1.	$\frac{\log 8}{\log 3} = 1.8928$	**16.**	8
2.	$\frac{\log \frac{1}{8}}{\log 5} = -1.2920$	**17.**	$\frac{1}{9}$
3.	$\frac{\log 24}{\log \frac{1}{2}} = 4.5850$	**18.**	7
4.	$2\log_7 x + 3\log_7 y + \log_7 z$	**19.**	$\frac{1}{3}$
5.	$2\log x - 3\log y - \frac{1}{2}\log z$	**20.**	2
6.	$3(2\log x + \log y + \log z)$	**21.**	-3
7.	$\log \frac{AB^2}{\sqrt[3]{C}} \quad A, B, C \neq 0$	**22.**	$\frac{k}{m}$
8.	$\log \frac{C^3}{D^4\sqrt[4]{E}} \quad C, D, E \neq 0$	**23.**	$\frac{3}{2}$
9.	$\log_2 \frac{F}{EK^3} \quad E, F, K \neq 0$	**24.**	$\frac{3}{2}$
10.	$\log_3 \frac{AC^3}{B^2\sqrt[4]{D}} \quad A, B, C, D \neq 0$	**25.**	$\frac{3}{2}\log_3(2)$
11.	$\log \frac{x^2}{y^3z^4} \quad x, y, z \neq 0$	**26.**	23
12.	x	**27.**	6
13.	y^2	**28.**	$\frac{a}{2}$
14.	8	**29.**	11
15.	7		



Topic 4 Logarithmic & Exponential Equations

There are three types of log equations you will run into.

1. log + log = log

example: $\log 2 + \log (x - 4) = \log 4$

2. log + log = #

example: $\log_2 (x + 5) - \log_2 3 = 2$

3. exponents that you cannot make the same base

example: $2^{x+1} = 7^{2x-3}$

as we did in
LG #3

EXAMPLE 1 - Solve Logarithmic Equations

Solve: $\log 2 + \log (x - 4) = \log 4$

1st ☛ Single log L.S. $\log 2x - 8 = \log 4$

2nd ☛ Cancel out logs on both sides $\cancel{\log} 2x - 8 = \cancel{\log} 4$

3rd ☛ Solve for x

$$
\begin{aligned}
2x - 8 &= 4 \\
2x &= 12 \\
x &= 6
\end{aligned}
$$

TRY: Solve:

a) $\log_5 (x - 2) - \log_5 3 = \log_5 2$

b) $\log (x - 1) + \log x = \log 12$

EXAMPLE 1b - Solve Logarithmic Equations

Solve: $\log_2 (x + 5) - \log_2 3 = 2$

1st ☛ Single log L.S. $\quad \log_2 \frac{x+5}{3} = 2$

2nd ☛ Boot-up log base $\quad \frac{x+5}{3} = 2^2$

3rd ☛ Solve for x $\quad \frac{x+5}{3} = 4$

$$x + 5 = 12$$

$$x = 7$$

TRY: Solve the following Logarithmic Equations.

a) $\log_3 (2x - 5) + \log_3 4 = 2$ b) $\log_6 (x + 3) + \log_6 (x - 2) = 1$

Watch out for Log Equations like this.

$$\log_5 (2 - x) + \log_5 2 = 1$$

EXAMPLE 2 - Solve Exponential Equations Using Logarithms

Solve: $2^{x+1} = 7^{2x-3}$

1st ☛ Log both sides

$$\log 2^{x+1} = \log 7^{2x-3}$$

$$(x + 1) \log 2 = (2x-3) \log 7$$

2nd ☛ Distribute

$$x\log 2 + \log 2 = 2x\log 7 - 3\log 7$$

3rd ☛ Collect like terms

$$x\log 2 - 2x\log 7 = -3\log 7 - \log 2$$

4th ☛ Factor out x

$$x (\log 2 - 2\log 7) = -3\log 7 - \log 2$$

5th ☛ Divide out logs

$$\frac{x\,\cancel{(\log 2 - 2\log 7)}}{\cancel{(\log 2 - 2\log 7)}} = \frac{-3\log 7 - \log 2}{(\log 2 - 2\log 7)}$$

$$x = \frac{-3\log 7 - \log 2}{\log 2 - 2\log 7} = \mathbf{2.04}$$

TRY: a) Solve exact. $2^x = 5$

TRY: b) Solve to 2 decimal places. $5^{3x+2} = 8^{x-3}$

Complete Assignment Questions #1 - #4

Assignment

1. Find the value of x to the nearest hundredth.

a) $4^x = 60$ **b)** $7^{x+2} = 41$ **c)** $4^{x+1} = 5^{x-2}$

3. Solve for the variable in each equation.

a) $\log_3 x - \log_3 3 = \log_3 30$ **b)** $\log_3 3y - \log_3 4 = \log_3 6$ **c)** $2 \log y = \log 25$

4. Solve for the variable in each equation.

a) $\log_5 x - \log_5 (x - 1) = \log_5 3$

b) $\log_4(x - 5) + \log_4(x - 2) = 1$

c) $\log_8(-x) + \log_8(3 - x) = \log_8 10$

d) $\log_3(3x - 1) - \log_3(x - 1) = 4$

Answer Key

1. **a)** 2.95 **b)** –0.09 **c)** 20.64

3. **a)** 90 **b)** 8 **c)** 5

4. **a)** $\frac{3}{2}$ **b)** 6 **c)** –2 **d)** $\frac{40}{39}$

LG 5 Worksheet C (Logarithmic Equations)

Solve for x exactly.

1. $\log(x) = 2$

2. $\log(x + 1) = -1$

3. $\log(\log x) = 1$

4. $\log(1000) = x$

5. $2\log_3(x) = 4$

6. $\log(2x) = \log(2) + \log(4)$

7. $\log(6) - \log(5) = \log(x)$

8. $\log(x) - \log(7) = \log(3)$

9. $\log_3(x) + \log_3(x) = \log_3(16)$

10. $\log(4) = x\log(2)$

11. $\log_9(16) = 4\log_9(x)$

12. $\log_2(x) + \log_2(3) = 4$

13. $\log_3(x) - \log_3(2) = 2$

14. $\log_2(\log_3(x)) = 2$

15. $\log_2 x + \log_2(x - 2) = \log_2 8$

16. $\log_2(x) + \log_2(x - 2) = 3$

17. $\log(x+2)+\log(x-1)=1$

18. $\log(3x+2)+\log(x-1)=2$

19. $\log_2(x)-\log_2(x-3)=4$

20. $\log_3(x-2)-\log_3(x+1)=\log_3 5$

21. $\log(3x^2+2x-4)=0$

22. $\log_5(x-3)+\log_5(x+4)-\log_5(x)=\log_5(5)$

Answer Key

1.	100	**16.**	4, reject -2
2.	$\frac{-9}{10}$	**17.**	3, reject -4
3.	10^{10}	**18.**	6, reject -5.667
4.	3	**19.**	$\frac{16}{5}$
5.	9	**20.**	No Solution, Reject $\frac{-7}{4}$
6.	4	**21.**	1, $\frac{-5}{3}$
7.	$\frac{6}{5}$	**22.**	6, reject -2
8.	21		
9.	4		
10.	2		
11.	2		
12.	$\frac{16}{3}$		
13.	18		
14.	81		
15.	4, reject -2		

PRE-CALCULUS 12

Seminar Notes

Learning Guides 6

GEOMETRIC SEQUENCES & SERIES

Types of Sequences

Sequences

Arithmetic Sequences ***Geometric Sequences*** ***Other Sequences***

Arithmetic Sequence - a sequence where each term is formed from the preceding term by ***adding*** a constant (positive or negative) to it., eg., 7, 11, 15, 19, ...

This constant is called the **common difference**.

The common difference in the example above is _____ .

Geometric Sequence - is a sequence where each term is obtained by ***multiplying*** the preceding term by a constant, eg., 2, 6, 18, 54, ...

This constant is called the **common ratio**.

The common ratio in the example above is _____ .

Other Sequences - sequences which are neither arithmetic nor geometric.

★USE FORMULA SHEET

Geometric Sequences

- The general term of a geometric sequence is:

$$\boxed{t_n = t_1(r)^{n-1}}$$

This is on your Formula Sheet

$t_1 =$

$r =$

$n =$

$t_n =$

Topic 1

Example 1

Write a General Term

a) Write a general term t_n for:

4, 20, 100, 500, ...

b) Write a general term t_n for:

1, -3, 9, -27, ...

Topic 2

Example 2

Find a Specific Term

a) Write a general term t_n for

$1, 5, 25, 125, ...t_{10}$

b) Write a general term t_n for

$2, 4, 8, 16, ...$

What is term t_{20}?

★ **WATCH OUT for t_0**

c) Suppose the smallest reduction a photo copier could make is 60% of the original. What is the shortest possible length after 8 reductions of a photograph that is originally 42 cm long?

Topic 3

Example 3

Find the Term Number

a) For 3, 6, 12, 24, ...which term is 3072?

b) For 2, 8, 32, 128, … , which term is 32768?

Topic 4

Example 4

Finding Specific Term(s)

a) What are the 2 geometric means between 2 and 250?

b) In a geometric sequence,
$t_3 = -12$ and $t_8 = -384$,
Find the first two terms.

C) In a geometric sequence, the second term is 28 and the fifth term is 1792.

Determine the value of t_1 and r, and list the first three terms of the sequence.

LG 6 Worksheet A (Geometric Sequences)

1. Determine if the sequence is geometric. If it is, state the common ratio and the general term in the form $t_n = t_1 r^{n-1}$.

a) 1, 2, 4, 8, …

b) 2, 4, 6, 8, …

c) 3, −9, 27, −81, …

d) 1, 1, 2, 4, 8, …

e) 10, 15, 22.5, 33.75, …

f) −1, −5, −25, −125, …

2. Copy and complete the following table for the given geometric sequences.

	Geometric Sequence	Common Ratio	6th Term	10th Term
a)	6, 18, 54, ...			
b)	1.28, 0.64, 0.32, ...			
c)	$\frac{1}{5}, \frac{3}{5}, \frac{9}{5}, \ldots$			

3. Determine the first four terms of each geometric sequence.

a) $t_1 = 2, r = 3$

b) $t_1 = -3, r = -4$

c) $t_1 = 4, r = -3$

d) $t_1 = 2, r = 0.5$

4. Determine the missing terms, t_2, t_3, and t_4, in the geometric sequence in which $t_1 = 8.1$ and $t_5 = 240.1$.

5. Determine a formula for the nth term of each geometric sequence.

a) $r = 2, t_1 = 3$

b) 192, −48, 12, −3, …

c) $t_3 = 5, t_6 = 135$

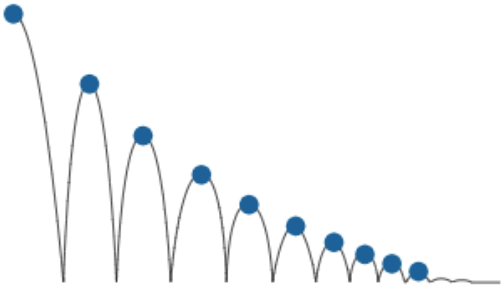
d) $t_1 = 4, t_{13} = 16\ 384$

6. Given the following geometric sequences, determine the number of terms, n.

Table A

	First Term, t_1	Common Ratio, r	nth Term, t_n	Number of Terms, n
a)	5	3	135	
b)	−2	−3	−1458	
c)	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{48}$	
d)	4	4	4096	
e)	$-\frac{1}{6}$	2	$-\frac{128}{3}$	
f)	$\frac{p^2}{2}$	$\frac{p}{2}$	$\frac{p^9}{256}$	

9. A ball is dropped from a height of 3.0 m. After each bounce it rises to 75% of its previous height.



a) Write the first term and the common ratio of the geometric sequence.

b) Write the general term for the sequence in part a).

c) What height does the ball reach after the 6th bounce?

d) After how many bounces will the ball reach a height of approximately 40 cm?

10. The colour of some clothing fades over time when washed. Suppose a pair of jeans fades by 5% with each washing.

a) What percent of the colour remains after one washing?

b) If $t_1 = 100$, what are the first four terms of the sequence?

c) What is the value of r for your geometric sequence?

d) What percent of the colour remains after 10 washings?

e) How many washings would it take so that only 25% of the original colour remains in the jeans? What assumptions did you make?

18. The Russian nesting doll or Matryoshka had its beginnings in 1890. The dolls are made so that the smallest doll fits inside a larger one, which fits inside a larger one, and so on, until all the dolls are hidden inside the largest doll. In a set of 50 dolls, the tallest doll is 60 cm and the smallest is 1 cm. If the decrease in doll size approximates a geometric sequence, determine the common ratio. Express your answer to three decimal places.

19. The primary function for our kidneys is to filter our blood to remove any impurities. Doctors take this into account when prescribing the dosage and frequency of medicine. A person's kidneys filter out 18% of a particular medicine every two hours.

a) How much of the medicine remains after 12 h if the initial dosage was 250 mL? Express your answer to the nearest tenth of a millilitre.

b) When there is less than 20 mL left in the body, the medicine becomes ineffective and another dosage is needed. After how many hours would this happen?

Answer Key

1. a) geometric; $r = 2$; $t_n = 2^{n-1}$
b) not geometric
c) geometric; $r = -3$; $t_n = 3(-3)^{n-1}$
d) not geometric
e) geometric; $r = 1.5$; $t_n = 10(1.5)^{n-1}$
f) geometric; $r = 5$; $t_n = -1(5)^{n-1}$

2.

	Geometric Sequence	Common Ratio	6th Term	10th Term
a)	6, 18, 54, ...	3	1458	118 098
b)	1.28, 0.64, 0.32, ...	0.5	0.04	0.0025
c)	$\frac{1}{5}, \frac{3}{5}, \frac{9}{5}, \ldots$	3	$\frac{243}{5}$	$\frac{19\,683}{5}$

3. a) 2, 6, 18, 54 **b)** −3, 12, −48, 192
c) 4, −12, 36, −108 **d)** $2, 1, \frac{1}{2}, \frac{1}{4}$

4. 18.9, 44.1, 102.9

5. a) $t_n = 3(2)^{n-1}$ **b)** $t_n = 192\left(-\frac{1}{4}\right)^{n-1}$
c) $t_n = \frac{5}{9}(3)^{n-1}$ **d)** $t_n = 4(2)^{n-1}$

6. a) 4 **b)** 7 **c)** 5
d) 6 **e)** 9 **f)** 8

7. 37

8. 16, 12, 9; $t_n = 16\left(\frac{3}{4}\right)^{n-1}$

9. a) $t_1 = 3$; $r = 0.75$
b) $t_n = 3(0.75)^{n-1}$
c) approximately 53.39 cm
d) 7

10. a) 95%
b) 100, 95, 90.25, 85.7375
c) 0.95
d) about 59.87%
e) After 27 washings, 25% of the original colour would remain in the jeans. Example: The geometric sequence continues for each washing.

18. 0.920
19. a) 76.0 mL **b)** 26 h

The SUM of a Geometric series can be determined using the formulas:

$$S_n = \frac{t_1(r^n - 1)}{r - 1} \quad , \quad r \neq 1$$

$$S_n = \frac{rt_n - t_1}{r - 1} \quad , \quad r \neq 1$$

☆ used if t_n is known

where: t_1 is first term

n is the number of terms

r is the common ratio

S_n is the sum of the first n terms

Topic 5

Finding the Sum of a Geometric Series

Example 1

Determine the sum of the first 12 terms of each geometric series

a) $5 + 10 + 20 + 40 + \ldots$ (S_{12})

Example 2

Given: $t_1 = \frac{1}{12}$, $r = -3$, $n = 8$. Find the S_n.

Express answer as exact values in fraction form.

Example 3

Find the number of terms in a geometric series when the sum is 2730 and the first two terms are 2, 8.

The SUM of an Infinite Geometric series can be determined using the formula:

$$S_\infty = \frac{t_1}{1-r}$$

☆ **Important to check to see that the Ratio is between -1 and 1.** ☆

Since -1 < r < 1, the series is convergent.

where: t_1 is first term
r is the common ratio
S_n is the sum of the first n terms

Topic 6

Finding the Sum of an Infinite Series

Example 1

Determine the *sum to infinity*.

i) $5 + \frac{5}{4} + \frac{5}{16} + \ldots$

ii) $t_1 = 4,\ r = -\frac{1}{3}$

Example 2

Find the ratio of infinite series that has a sum of 10 and the first term is $\frac{1}{2}$.

LG 6 Worksheet B (Geometric Series)

1. Determine whether each series is geometric. Justify your answer.

a) $4 + 24 + 144 + 864 + \cdots$

b) $-40 + 20 - 10 + 5 - \cdots$

c) $3 + 9 + 18 + 54 + \cdots$

d) $10 + 11 + 12.1 + 13.31 + \cdots$

2. For each geometric series, state the values of t_1 and r. Then determine each indicated sum. Express your answers as exact values in fraction form and to the nearest hundredth.

a) $6 + 9 + 13.5 + \cdots (S_{10})$

b) $18 - 9 + 4.5 + \cdots (S_{12})$

c) $2.1 + 4.2 + 8.4 + \cdots (S_9)$

d) $0.3 + 0.003 + 0.000\ 03 + \cdots (S_{12})$

3. What is S_n for each geometric series described? Express your answers as exact values in fraction form.

a) $t_1 = 12, r = 2, n = 10$

b) $t_1 = 27, r = \frac{1}{3}, n = 8$

c) $t_1 = \frac{1}{256}, r = -4, n = 10$

d) $t_1 = 72, r = \frac{1}{2}, n = 12$

4. Determine S_n for each geometric series. Express your answers to the nearest hundredth, if necessary.

a) $27 + 9 + 3 + \cdots + \frac{1}{243}$

b) $\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots + \frac{128}{6561}$

c) $t_1 = 5, t_n = 81\ 920, r = 4$

d) $t_1 = 3, t_n = 46\ 875, r = -5$

5. What is the value of the first term for each geometric series described? Express your answers to the nearest tenth, if necessary.

a) $S_n = 33, t_n = 48, r = -2$

b) $S_n = 443, n = 6, r = \frac{1}{3}$

6. The sum of $4 + 12 + 36 + 108 + \cdots + t_n$ is 4372. How many terms are in the series?

7. The common ratio of a geometric series is $\frac{1}{3}$ and the sum of the first 5 terms is 121.

a) What is the value of the first term?

b) Write the first 5 terms of the series.

8. What is the second term of a geometric series in which the third term is $\frac{9}{4}$ and the sixth term is $-\frac{16}{81}$? Determine the sum of the first 6 terms. Express your answer to the nearest tenth.

10. A tennis ball dropped from a height of 20 m bounces to 40% of its previous height on each bounce. The total vertical distance travelled is made up of upward bounces and downward drops. Draw a diagram to represent this situation. What is the total vertical distance the ball has travelled when it hits the floor for the sixth time? Express your answer to the nearest tenth of a metre.

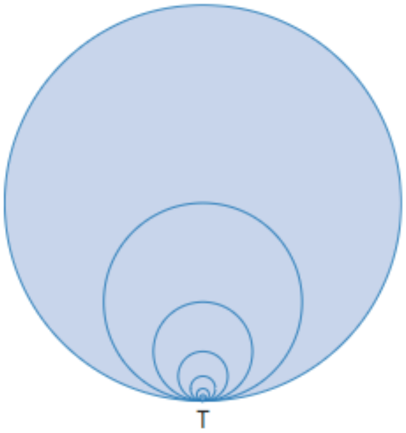
16. Determine the number of terms, n, if $3 + 3^2 + 3^3 + \cdots + 3^n = 9840$.

17. The third term of a geometric series is 24 and the fourth term is 36. Determine the sum of the first 10 terms. Express your answer as an exact fraction.

18. Three numbers, a, b, and c, form a geometric series so that $a + b + c = 35$ and $abc = 1000$. What are the values of a, b, and c?

19. The sum of the first 7 terms of a geometric series is 89, and the sum of the first 8 terms is 104. What is the value of the eighth term?

20. A fractal is created as follows: A circle is drawn with radius 8 cm. Another circle is drawn with half the radius of the previous circle. The new circle is tangent to the previous circle at point T as shown. Suppose this pattern continues through five steps. What is the sum of the areas of the circles? Express your answer as an exact fraction.



Answer Key

1. a) geometric; $r = 6$ **b)** geometric; $r = -\frac{1}{2}$

c) not geometric **d)** geometric; $r = 1.1$

2. a) $t_1 = 6$, $r = 1.5$, $S_{10} = \frac{174\,075}{256}$, $S_{10} \approx 679.98$

b) $t_1 = 18$, $r = -0.5$, $S_{12} = \frac{12\,285}{1024}$, $S_{12} \approx 12.00$

c) $t_1 = 2.1$, $r = 2$, $S_9 = \frac{10\,731}{10}$, $S_9 = 1073.10$

d) $t_1 = 0.3$, $r = 0.01$, $S_{12} = \frac{10}{33}$, $S_{12} \approx 0.30$

3. a) 12 276 **b)** $\frac{3280}{81}$

c) $-\frac{209\,715}{256}$ **d)** $\frac{36\,855}{256}$

4. a) 40.50 **b)** 0.96

c) 109 225 **d)** 39 063

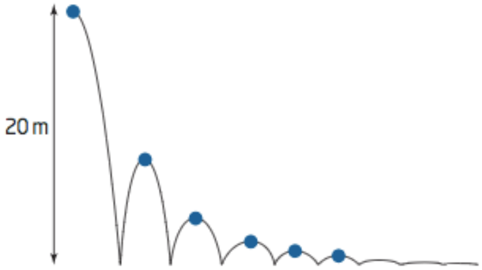
5. a) 3 **b)** 295.7

6. 7

7. a) 81 **b)** $81 + 27 + 9 + 3 + 1$

8. $t_2 = -\frac{81}{16}$; $S_6 = 7.8$

10. 46.4 m



16. 8

17. $\frac{58\,025}{48}$

18. $a = 5$, $b = 10$, $c = 20$ or $a = 20$, $b = 10$, $c = 5$

19. 15

20. $\frac{341}{4}\pi$

LG 6 Worksheet C (Infinite Geometric Series)

2. Determine the sum of each infinite geometric series, if it exists.

a) $t_1 = 8, r = -\frac{1}{4}$

b) $t_1 = 3, r = \frac{4}{3}$

c) $t_1 = 5, r = 1$

d) $1 + 0.5 + 0.25 + \cdots$

e) $4 - \frac{12}{5} + \frac{36}{25} - \frac{108}{125} + \cdots$

3. Express each of the following as an infinite geometric series. Determine the sum of the series.

a) $0.\overline{87}$

b) $0.\overline{437}$

6. The sum of an infinite geometric series is 81, and its common ratio is $\frac{2}{3}$. What is the value of the first term? Write the first three terms of the series.

7. The first term of an infinite geometric series is -8, and its sum is $-\frac{40}{3}$. What is the common ratio? Write the first four terms of the series.

8. In its first month, an oil well near Virden, Manitoba produced 24 000 barrels of crude. Every month after that, it produced 94% of the previous month's production.

a) If this trend continued, what would be the lifetime production of this well?

b) What assumption are you making? Is your assumption reasonable?

Answer Key

2. a) $\frac{32}{5}$ **b)** no sum

c) no sum **d)** 2

e) 2.5

3. a) $0.87 + 0.0087 + 0.000\,087 + \cdots;$

$S_\infty = \frac{87}{99}$ or $\frac{29}{33}$

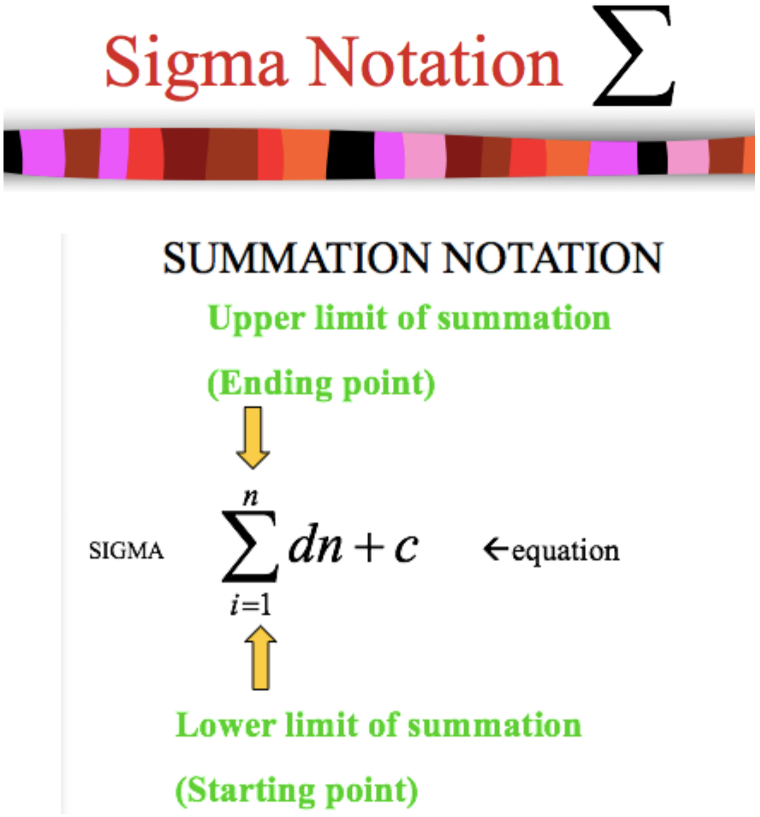
b) $0.437 + 0.000\,437 + \cdots; S_\infty = \frac{437}{999}$

6. $t_1 = 27; 27 + 18 + 12 + \cdots$

7. $r = \frac{2}{5}; -8 - \frac{16}{5} - \frac{32}{25} - \frac{64}{125} - \cdots$

8. a) 400 000 barrels of oil

b) Determining the lifetime production assumes the oil well continues to produce at the same rate for many months. This is an unreasonable assumption because 94% is a high rate to maintain.



Sigma Notation $\sum$

SUMMATION NOTATION

Upper limit of summation

(Ending point)

SIGMA $\sum_{i=1}^{n} dn + c$ ←equation

Lower limit of summation

(Starting point)

Topic 1 **Example 1**

Write out each of the following sums.

$$\sum_{n=0}^{3}(2n+1) = 16$$

$$= [2(0) + 1] + [2(1) + 1] + [2(2) + 1] + [2(3) + 1]$$

$$= 1+3+5+7$$

$$\boxed{=16}$$

You Try:

a)

$$\sum_{k=3}^{6}(1+k^2)$$

b)

$$\sum_{n=1}^{5}n^2$$

c)

$$\sum_{i=1}^{5}-3i$$

Topic 2 Example 1

Write each series using sigma notation.

a)

$3 + 5 + 7 + 9$

b)

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}+\frac{6}{7}+\frac{7}{8}+\frac{8}{9}$$

c)

1, 3, 9, 27, 81

d)

$$2, \frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \frac{1}{128} \ldots \frac{1}{8192}$$

You Try:

a) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25} \frac{1}{36}, \frac{1}{49}$

b) -2, 6, -18, 54, -162…4374

LG 6 Worksheet D (Sigma Notation)

Write each series using sigma notation.

1. $3 + 5 + 7 + 9 + 11 + 13$

2. $\frac{1}{2} + \frac{1}{6} + \frac{1}{18} + \frac{1}{54}$

3. $1 + 4 + 9 + 16 + 25 + 36 + 49 + 64$

4. $100 + 50 + 25 + \ldots$

Write out each of the following sums.

5. $\sum_{k=1}^{7} 4^{k-1}$

6. $\sum_{n=2}^{8} (-6)^{n-1}$

7. $\sum_{n=1}^{8} 2(-2)^{n-1}$

8. $\sum_{n=6}^{9} 4(3)^{n-1}$

Write each series using sigma notation.

9. $-2 + 1 + 4 + \ldots + 13$

10. $81 + 27 + \ldots + \frac{1}{3}$

11. $\frac{1}{8} + \frac{1}{4} + \ldots + 64$

12. $3 + 11 + 19 + \ldots + 787$

13. $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots + \frac{1}{100}$

14. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots + \frac{99}{100}$

Answer Key

1. $\sum_{n=1}^{6} 2n + 1$

2. $\sum_{n=1}^{4} \frac{1}{2}\left(\frac{1}{3}\right)^{n-1}$

3. $\sum_{n=1}^{8} n^2$

4. $\sum_{n=1}^{\infty} 100\left(\frac{1}{2}\right)^{n-1}$

5. $1 + 4 + 16 + 64 + 256 + 1024 + 4096$

6. $-6 + 36 - 216 + 1296 - 7776 + 46656 - 279936$

7. $2 - 4 + 8 - 16 + 32 - 64 + 128 - 256$

8. $972 + 2916 + 8748 + 26244$

9. $\sum_{n=1}^{6} 3n - 5$

10. $\sum_{n=1}^{6} 81\left(\frac{1}{3}\right)^{n-1}$

11. $\sum_{n=1}^{10} \frac{1}{8}(2)^{n-1}$

12. $\sum_{n=1}^{99} 8n - 5$

13. $\sum_{n=2}^{99} \frac{1}{n+1}$

14. $\sum_{n=1}^{99} \frac{n}{n+1}$

LG6 Quiz (Geometric Sequence & Series)

1. Determine if the sequence is geometric. If it is, find the common ratio, the 8th term, and the explicit formula.

$a)\ -1, -3, -9, -27, \ldots$

$b)\ 2, \frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \ldots$

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2. Find the missing term or terms in each geometric sequence.

a) **…, 4, ____, ____, 108, …**

b) **…, -25, ____, ____, ____,** $-\frac{1}{25}$**, …**

---

3. Find the first term of the geometric sequence that has a fourth term of 40 and a common ratio of 2.

4. Find the sum of each geometric series.

$a)\ find\ s_6\ \ for\ \ 2, -6,\ 18, \ldots$ $\qquad b)\ 30 + 20 + \frac{40}{3} + \cdots + \frac{1280}{729}$

---

5. Suppose that five people are ill during the first week of an epidemic and each sick person spreads the contagious disease to four other people by the end of the second week and so on. By the end of the 15th week, how many people will be affected by the epidemic.

---

6. A gardener wanted to reward a worker for their good deeds by giving them some apples. The gardener gave the worker two choices. You can either have 1000 apples at once or you could get 1 apple on the first day, 2 apples on the second day, 4 on the third day, 8 on the fourth day and so on for the next ten days. Which option gets the maximum apples?

*****SEE YOUR TEACHER FOR MARKING KEY*****

PRE-CALCULUS 12

Seminar Notes

Learning Guides 7 & 8

FUNCTION OPERTIONS & RATIONAL FUNCTION

Notation for Function Operations

Operation	Notation
Addition	$(f+g)(x) = f(x) + g(x)$
Subtraction	$(f-g)(x) = f(x) - g(x)$
Multiplication	$(fg)(x) = f(x) \cdot g(x)$
Division	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$

Topic 1 SUM AND DIFFERENCES OF FUNCTIONS

EXAMPLE 1: Determine the Sum of Two Functions $(f+g)(x)$ $\quad where: \; f(x) = 2x - 1$

$$g(x) = 3 - x$$

a) Determine the equation of the function.

$$h(x) = (f+g)(x) \longrightarrow f(x) + g(x)$$

$$(2x-1) + (3-x)$$

Collect Like Terms

$$2x - x - 1 + 3$$

$$h(x) = x + 2$$

b) State the domain and range of $h(x)$.

Since the domain and range for both f(x) and g(x) are all real numbers, the domain and range for h(x) is the same. $\{x \mid x \in R\}$ $\{y \mid y \in R\}$

EXAMPLE 2: Determine the Difference of Two Functions

Consider the functions $f(x)=\sqrt{x-1}$ *and* $g(x)=x-2$

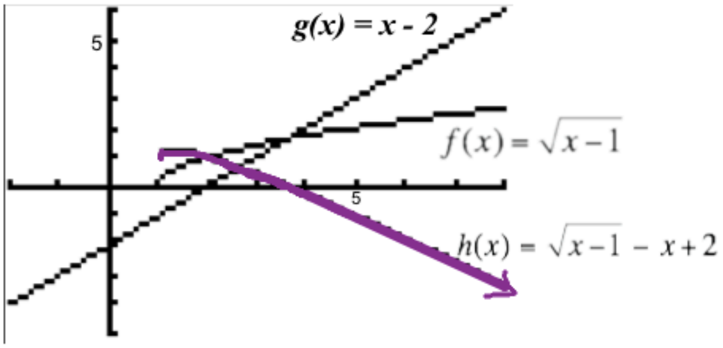
a) Determine the equation of the function.

$$h(x)=(f-g)(x)$$

$$\begin{aligned} h(x) &= f(x)-g(x) \\ &= \sqrt{x-1}-(x-2) \\ &= \sqrt{x-1}-x+2 \end{aligned}$$

b) State the domain and the range of *h(x)*. **See below**

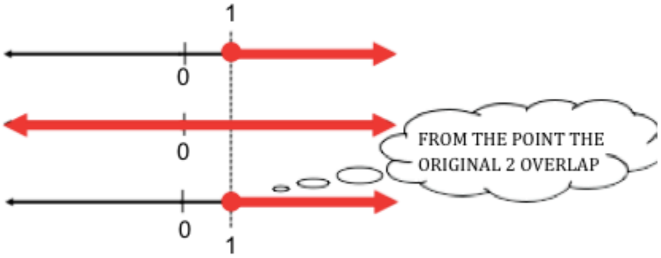
Graph each function to determine domain and range of *h(x)*.



Domain of $f(x)=\sqrt{x-1}$ $x \ge 1$

Domain of $g(x)=x-2$ *all R*

Domain of $h(x)=\sqrt{x-1}-x+2$ is $x \ge 1$



The range from looking at the graph is approximately

$y \le 1.25$

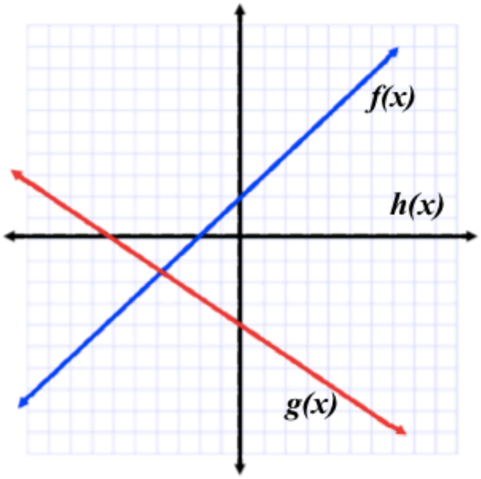
TRY: Consider the functions $f(x) = -4x + 3$ and $g(x) = 2x^2$.

a) Determine the equation of the function
$h(x) = (f + g)(x)$ and $m(x) = (g - f)(x)$.

b) State the domain and range for each.

EXAMPLE 3: Determine a Combined Function From Graphs.

Determine and sketch the graph of $h(x) = (f + g)(x)$ given the graphs of $f(x)$ and $g(x)$.



1st - find the equation for each function.

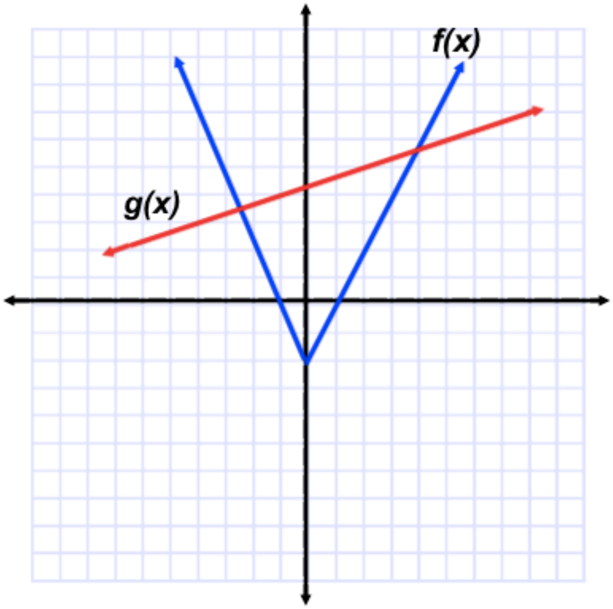
$y = mx + b$

f(x)	g(x)

2nd - add the two functions together as you did earlier.

EXAMPLE 3b: Use the graphs of $f(x)$ and $g(x)$ to evaluate the following:

a) $(f + g)(2)$ b) $(f - g)(-1)$



1st - draw a line where $x = 2$

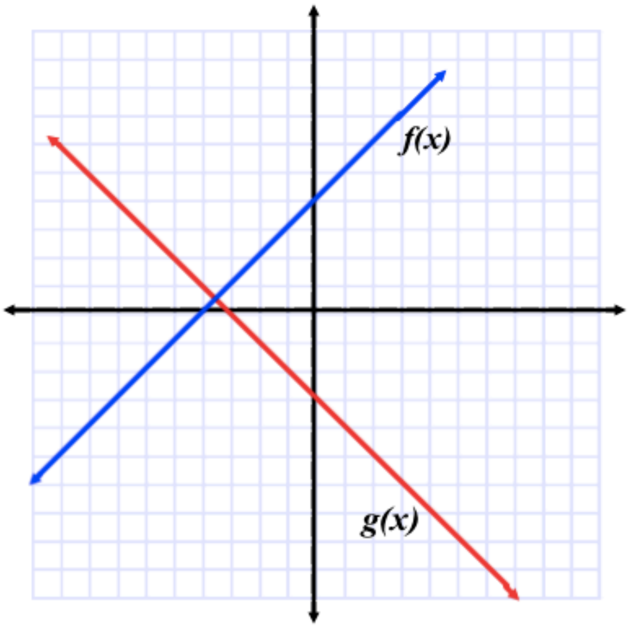
Why? Because you probably noticed that x was replaced with 2. $(f + g)(x) \rightarrow \underbrace{(f + g)(2)}_{\text{you see}}$

2nd - find the points where this line intersects the two given function f(x) and g(x).

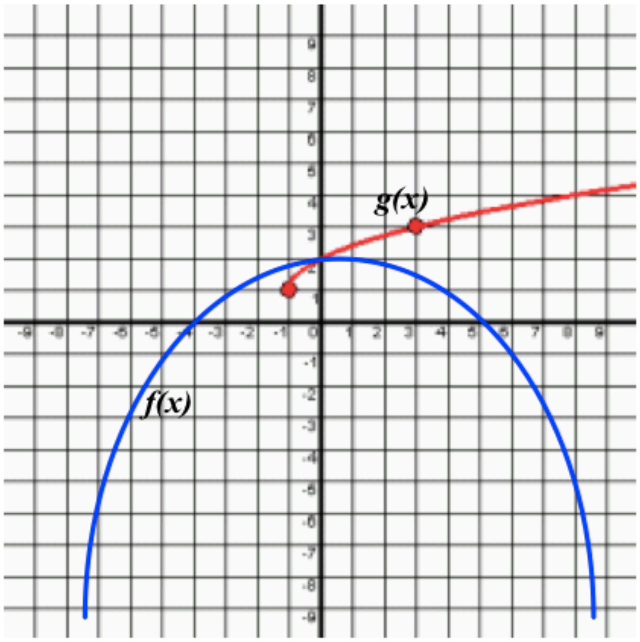
3rd - apply the operation

Repeat steps for b)

Try: 1. Determine and sketch the graph of $h(x) = (f - g)(x)$ given the graphs of $f(x)$ and $g(x)$.



Try 2. Use the graphs of $f(x)$ and $g(x)$ to evaluate the following $(f + g)(8)$.



EXAMPLE 4: Determine a Combined Function From Graphs.

Suppose the cost of T-shirts for a Math Conference includes $125 fixed cost and $7.50 per T-shirt. The T-shirts are sold for $12.00 each.

a) Write an equation to represent

- the total cost *C*, as a function of the number, *n*, of T-shirts produced
- the revenue, *R*, as a function of the number, *n*, of T-shirts sold

b) Graph the total cost and revenue functions on the same set of axes. What does the point of intersection represent?

c) Profit, *P*, is the difference between revenue and cost. Write a function representing *P* in terms of *n*.

SOLUTION:

a) Write an equation to represent

- the total cost *C*, as a function of the number, *n*, of T-shirts produced
- the revenue, *R*, as a function of the number, *n*, of T-shirts sold

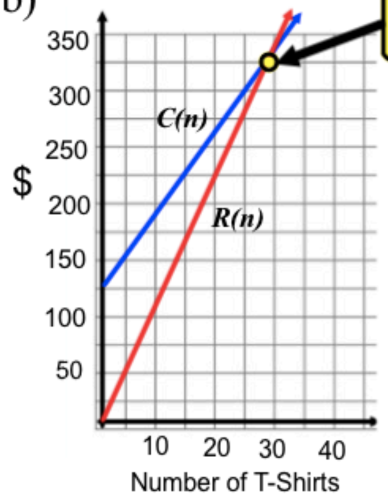
Solution

$C(n) = 7.5n + 125$

$R(n) = 12n$

Suppose the cost of T-shirts for a Math Conference includes $125 fixed cost and $7.50 per T-shirt. The T-shirts are sold for $12.00 each.

b)



This point represents break even point

c) $P(n) = R(n) - C(n)$

$P(n) = 12n - (7.5n + 125)$

$P(n) = 4.5n - 125$

Try: Suppose the cost for preparing booklets for a Math Conference includes \$675 in fixed cost and \$3.50 per booklet. The booklets are sold for \$30 each.

---

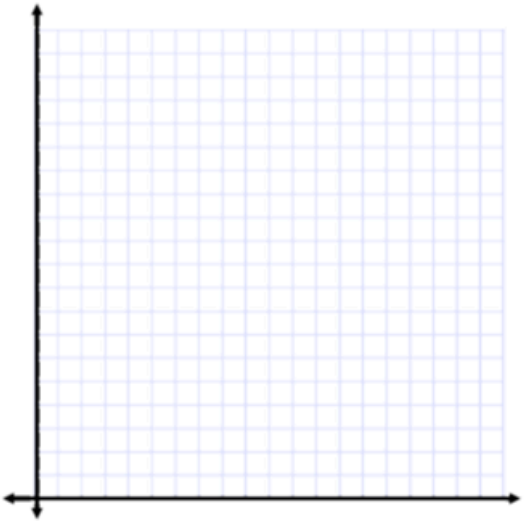
a) Write an equation to represent

- the total cost C, as a function of the number, n, of Booklets produced
- the revenue, R, as a function of the number, n, of Booklets sold

b) Graph the total cost and revenue functions on the same set of axes. What does the point of intersection represent?

c) Profit, P, is the difference between revenue and cost. Write a function representing P in terms of n.

Try solution:



Complete Assignment Questions #1 - #13

Assignment

1. For each pair of functions, determine $h(x) = f(x) + g(x)$.

a) $f(x) = |x - 3|$ and $g(x) = 4$

c) $f(x) = x^2 + 2x$ and $g(x) = x^2 + x + 2$

2. For each pair of functions, determine $h(x) = f(x) - g(x)$.

a) $f(x) = 6x$ and $g(x) = x - 2$

c) $f(x) = 6 - x$ and $g(x) = (x + 1)^2 - 7$

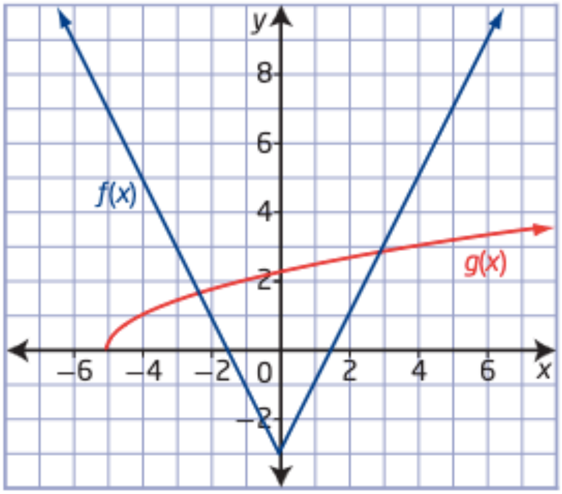
3. Consider $f(x) = -6x + 1$ and $g(x) = x^2$.

a) Determine $h(x) = f(x) + g(x)$ and find $h(2)$.

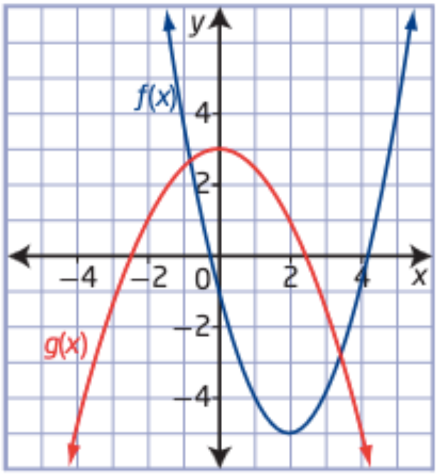
c) Determine $p(x) = g(x) - f(x)$ and find $p(1)$.

6. Use the graphs of $f(x)$ and $g(x)$ to evaluate the following.

a) $(f + g)(4)$ **b)** $(f + g)(-4)$



7. Use the graphs of $f(x)$ and $g(x)$ to determine which graph matches each combined function.

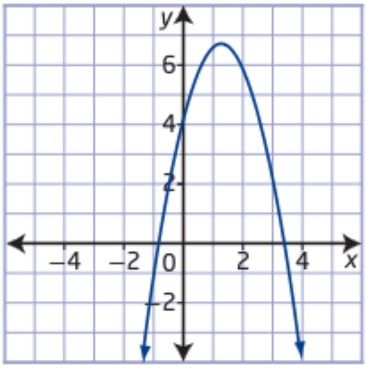


a) $y = (f + g)(x)$

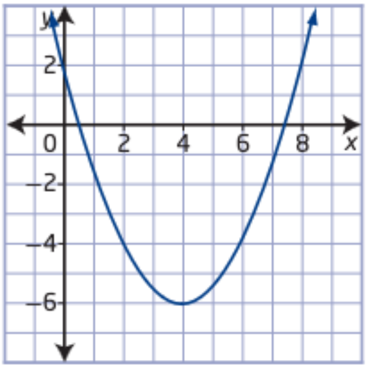
b) $y = (f - g)(x)$

c) $y = (g - f)(x)$

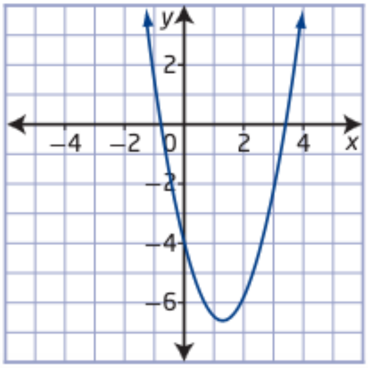
A



B



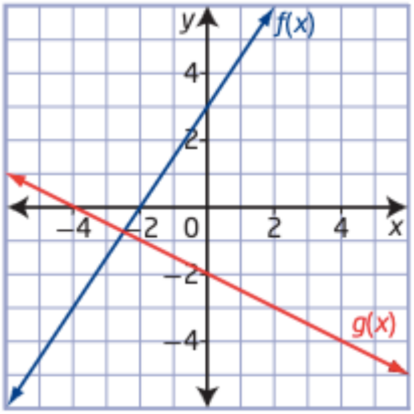
C



8. Copy each graph. Add the sketch of the graph of each combined function to the same set of axes.

i) $y = (f + g)(x)$
ii) $y = (f - g)(x)$
iii) $y = (g - f)(x)$

a)



13. The daily costs for a hamburger vendor are \$135 per day plus \$1.25 per hamburger sold. He sells each burger for \$3.50, and the maximum number of hamburgers he can sell in a day is 300.

a) Write equations to represent the total cost, C, and the total revenue, R, as functions of the number, n, of hamburgers sold.

b) Graph $C(n)$ and $R(n)$ on the same set of axes.

c) The break-even point is where $C(n) = R(n)$. Identify this point.

d) Develop an algebraic and a graphical model for the profit function.

e) What is the maximum daily profit the vendor can earn?

Answer Key

1. a) $h(x) = |x - 3| + 4$

c) $h(x) = 2x^2 + 3x + 2$

2. a) $h(x) = 5x + 2$

c) $h(x) = -x^2 - 3x + 12$

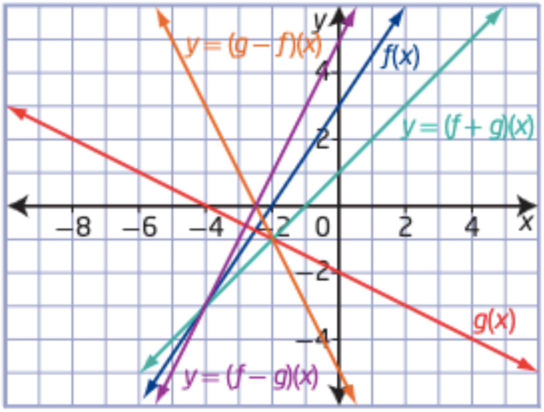
3. a) $h(x) = x^2 - 6x + 1$; $h(2) = -7$

c) $p(x) = x^2 + 6x - 1$; $p(1) = 6$

6. a) 8 **b)** 6

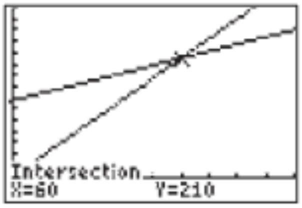
7. a) B **b)** C **c)** A

8. a)



13. a) $C(n) = 1.25n + 135$, $R(n) = 3.5n$

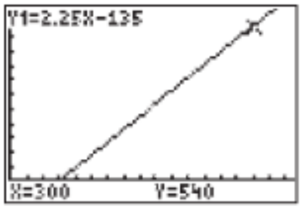
b)



```
Intersection
X=60        Y=210
```

c) (60, 210)

d) $P(n) = 2.25n - 135$



```
Y1=2.25X-135
X=300       Y=540
```

e) \$540

Topic 2

PRODUCTS AND QUOTIENTS OF FUNCTIONS

EXAMPLE 1: Determine the Product of Functions

Given $f(x) = (x + 2)^2 - 5$ and $g(x) = 3x - 4$

a) Determine the equation of the function.

$$h(x) = (f \bullet g)(x)$$

multiplication sign

Solution: $h(x) = (f(x))(g(x))$

$$((x + 2)^2 - 5)(3x - 4)$$

$$(x^2 + 4x + 4 - 5)(3x - 4)$$

$$3x^3 - 4x^2 + 12x^2 - 16x - 3x + 4$$

$$h(x) = 3x^3 + 8x^2 - 19x + 4$$

b) State domain and range of $h(x)$.

Solution: $h(x)$ formed a cubic functions thus, the domain and range are al Real Numbers.

$\{x \mid x \in R\}$ $\{y \mid y \in R\}$

EXAMPLE 2: Determine the Quotient of Functions

$$\left(\frac{f}{g}\right)(x) \quad where: \quad f(x) = x^2 - 4$$

$$g(x) = 2x - 4$$

a) Determine the equation of the function $h(x) = \frac{f(x)}{g(x)}$

When you have a denominator with a variable remember **non-permissibles!** **[restrictions]**

$2x$ - 4 is the denominator

To find non-permissibles make it $= 0$ and solve.

$$2x - 4 = 0$$

$$2x = 4$$

$$x = 2 \quad therefore \quad restriction \ is \quad x \neq 2$$

EXAMPLE 2b: Determine the Quotient of Functions

Given $f(x) = \dfrac{1}{x-2}$ and $g(x) = \dfrac{1}{x^2-4}$ find the equation of $\left(\dfrac{f}{x}\right)(x)$ and state its domain and range.

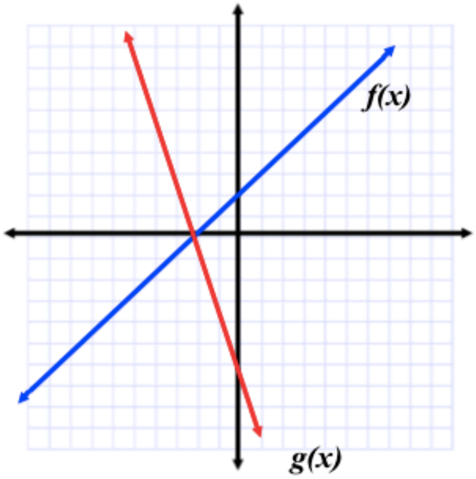
Try: Consider the functions $f(x) = x + 3$ and $g(x) = 2x^2 + 5x - 3$

a) Determine the equation of the function
$h(x) = (g \bullet f)(x)$ and $m(x) = \left(\dfrac{f}{g}\right)(x)$.

b) State the domain and range for each.

EXAMPLE 3: Determine a Combined Function From Graphs.

Determine and sketch the graph of h(x) = f(x)g(x) given the graphs of f(x) and g(x).



1st - find the equation for each function.

$y = mx + b$

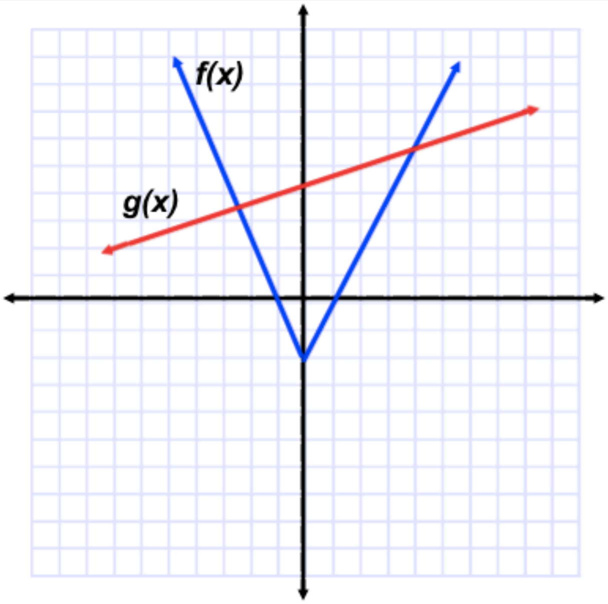
f(x)	*g(x)*

2nd - multiply the two functions together as you did earlier.

f(x) *g(x)*

EXAMPLE 3b:

Use the graphs of f(x) and g(x) to evaluate the following: a) $\frac{g(x)}{f(x)}(2)$ b) $f(x)g(x)(-1)$



1st - draw a line where $x = 2$

2nd - find the points where this line intersects the two given function *f(x)* and *g(x)*.

3rd - apply the operation

Repeat steps for b)

Complete Assignment Questions #1 - #5

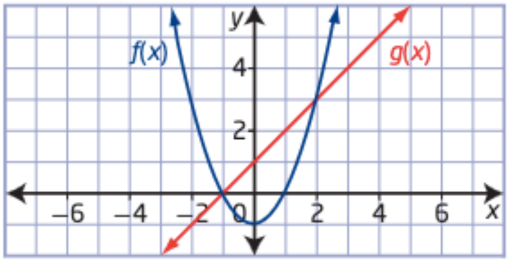
Assignment

1. Determine $h(x) = f(x)g(x)$ and $k(x) = \dfrac{f(x)}{g(x)}$ for each pair of functions.

a) $f(x) = x + 7$ and $g(x) = x - 7$

c) $f(x) = \sqrt{x + 5}$ and $g(x) = x + 2$

2. Use the graphs of $f(x)$ and $g(x)$ to evaluate the following.



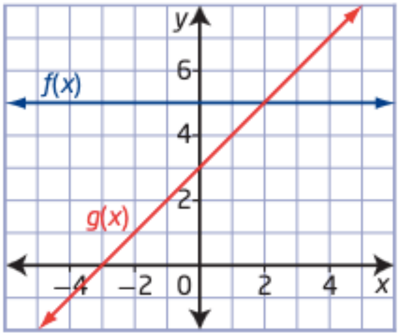
a) $(f \cdot g)(-2)$

b) $(f \cdot g)(1)$

c) $\left(\dfrac{f}{g}\right)(0)$

d) $\left(\dfrac{f}{g}\right)(1)$

3. Copy the graph. Add the sketch of the graph of each combined function to the same set of axes.



a) $h(x) = f(x)g(x)$

b) $h(x) = \dfrac{f(x)}{g(x)}$

4. For each pair of functions, $f(x)$ and $g(x)$,

- determine $h(x) = (f \cdot g)(x)$
- sketch the graphs of $f(x)$, $g(x)$, and $h(x)$ on the same set of coordinate axes
- state the domain and range of the combined function $h(x)$

a) $f(x) = x^2 + 5x + 6$ and $g(x) = x + 2$

b) $f(x) = x - 3$ and $g(x) = x^2 - 9$

c) $f(x) = \frac{1}{x + 1}$ and $g(x) = \frac{1}{x}$

5. Repeat #4 using $h(x) = \left(\frac{f}{g}\right)(x)$.

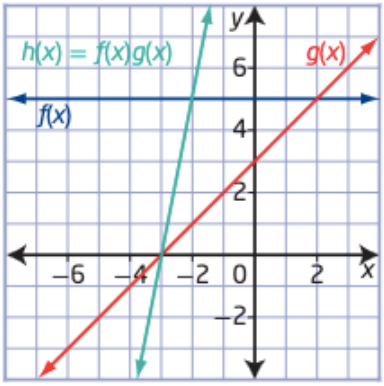
Answer Key

1. a) $h(x) = x^2 - 49$, $k(x) = \frac{x + 7}{x - 7}$, $x \neq 7$

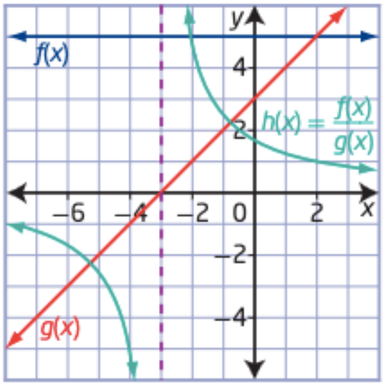
c) $h(x) = (x + 2)\sqrt{x + 5}$, $k(x) = \frac{\sqrt{x + 5}}{x + 2}$, $x \geq -5$, $x \neq -2$

2. a) -3 **b)** 0 **c)** -1 **d)** 0

3. a)



b)



4. a) $h(x) = x^3 + 7x^2 + 16x + 12$

domain $\{x \mid x \in R\}$, range $\{y \mid y \in R\}$

b) $h(x) = x^3 - 3x^2 - 9x + 27$

domain $\{x \mid x \in R\}$, range $\{y \mid y \in R\}$

c) $h(x) = \frac{1}{x^2 + x}$

domain $\{x \mid x \neq 0, -1, x \in R\}$,
range $\{y \mid y \leq -4 \text{ or } y > 0, y \in R\}$

5. a) $h(x) = x + 3$, $x \neq -2$

domain $\{x \mid x \neq -2, x \in R\}$,
range $\{y \mid y \neq 1, y \in R\}$

b) $h(x) = \frac{1}{x + 3}$, $x \neq \pm 3$

domain $\{x \mid x \neq \pm 3, x \in R\}$,
range $\left\{y \mid y \neq 0, \frac{1}{6}, y \in R\right\}$

c) $h(x) = \frac{x}{x + 1}$, $x \neq -1, 0$

domain $\{x \mid x \neq -1, 0, x \in R\}$,
range $\{y \mid y \neq 0, 1, y \in R\}$

Topic 3 COMPOSITE FUNCTIONS

EXAMPLE 1: Evaluate a Composite Function

If $f(x) = 2x + 1$ and $g(x) = 4 - x$, determine the value for $f(g(6))$.

EXAMPLE 2: Compose Functions With Restrictions

Consider $f(x) = \sqrt{x - 1}$ and $g(x) = x^2$.

a) Determine $(f \circ g)(x)$

This can be written as ***f(g(x))***

b) State the domain of $f(x)$, $g(x)$, and $(f \circ g)(x)$.

Solution:

1st substitute $g(x)$ into bracket $f(g(x)) = f(x^2)$

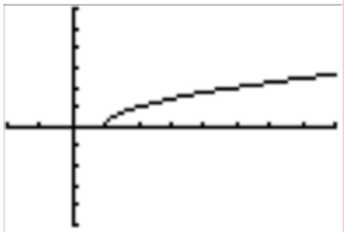
2nd now that you have taken care of the $g(x)$ write in the $f(x)$ $f(x^2) = \sqrt{x - 1}$

3rd now substitute (x^2) into the x and solve. $= \sqrt{x^2 - 1}$

$$\boldsymbol{(f \circ g)(x) = \sqrt{x^2 - 1}}$$

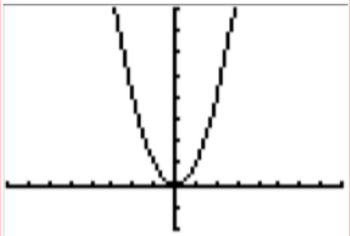
Solution: To find the domain graph each function.

$f(x)$



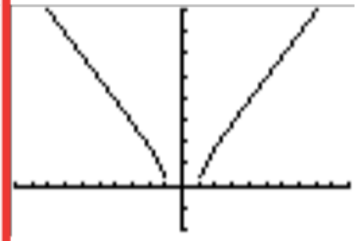
$D: \ x \geq 1$

$g(x)$



$D: \ x \in \mathrm{R}$

$(f \circ g)(x)$



$D: x \leq -1$ *or* $x \geq 1$

EXAMPLE 3: Determine the Composition of Two Functions

If $f(x) = x^2 + 3$ and $g(x) = x - 1$, determine the value of $f(g(x))$.

EXAMPLE 3b: Determine the Composition of Two Functions

If $f(x) = x^2 + 3$ and $g(x) = x - 1$, determine the value of $f(g(-2))$.

Try: 1. If $f(x) = |x|$ and $g(x) = x + 1$, determine $f(g(-11))$.

2. Consider $f(x) = \sqrt{x - 1}$ and $g(x) = -x^2$.

a) Determine $(g \circ f)(x)$
b) State the domain of $f(x)$, $g(x)$, and $(g \circ f)(x)$.

3. Given $f(x) = x^2 + 3x - 4$ and $g(x) = x + 1$, determine $f(g(x))$, and state domain and range.

EXAMPLE 4: Determine the Original Functions From a Composition

If $h(x) = f(g(x))$, determine $f(x)$ and $g(x)$ where

$$h(x) = (x - 2)^2 + (x - 2) + 1$$

TRY: If $h(x) = f(g(x))$, determine $f(x)$ and $g(x)$

a) $h(x) = \sqrt{x^2 + 1}$

Complete Assignment Questions #1 - #7

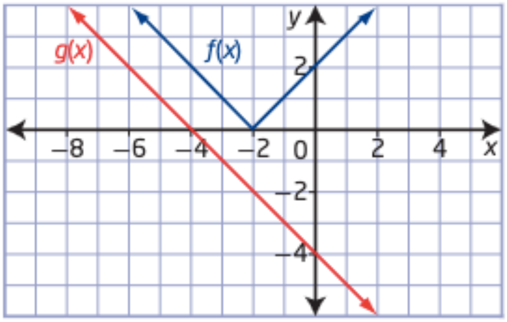
Assignment

1. Given $f(2) = 3$, $f(3) = 4$, $f(5) = 0$, $g(2) = 5$, $g(3) = 2$, and $g(4) = -1$, evaluate the following.

a) $f(g(3))$ **b)** $f(g(2))$

c) $g(f(2))$ **d)** $g(f(3))$

2. Use the graphs of $f(x)$ and $g(x)$ to evaluate the following.



a) $f(g(-4))$ **b)** $f(g(0))$

c) $g(f(-2))$ **d)** $g(f(-3))$

3. If $f(x) = 2x + 8$ and $g(x) = 3x - 2$, determine each of the following.

a) $f(g(1))$ **b)** $f(g(-2))$

c) $g(f(-4))$ **d)** $g(f(1))$

5. For each pair of functions, $f(x)$ and $g(x)$, determine $f(g(x))$ and $g(f(x))$.

a) $f(x) = x^2 + x$ and $g(x) = x^2 + x$

b) $f(x) = \sqrt{x^2 + 2}$ and $g(x) = x^2$

c) $f(x) = |x|$ and $g(x) = x^2$

7. If $h(x) = (f \circ g)(x)$, determine $g(x)$.

a) $h(x) = (2x - 5)^2$ and $f(x) = x^2$

b) $h(x) = (5x + 1)^2 - (5x + 1)$ and $f(x) = x^2 - x$

Answer Key

1. a) 3 **b)** 0 **c)** 2 **d)** −1

2. a) 2 **b)** 2 **c)** −4 **d)** −5

3. a) 10 **b)** −8 **c)** −2 **d)** 28

5. a) $f(g(x)) = x^4 + 2x^3 + 2x^2 + x$,
$g(f(x)) = x^4 + 2x^3 + 2x^2 + x$

b) $f(g(x)) = \sqrt{x^4 + 2}$, $g(f(x)) = x^2 + 2$

c) $f(g(x)) = x^2$, $g(f(x)) = x^2$

7. a) $g(x) = 2x - 5$ **b)** $g(x) = 5x + 1$

LG 7 Worksheet (Function Operation)

Perform the indicated operation.

1.
$g(t) = 2t + 5$
$f(t) = -t^2 + 5$
Find $(g + f)(t)$

2.
$h(n) = 4n + 5$
$g(n) = 3n + 4$
Find $(h - g)(n)$

3.
$f(x) = 3x - 1$
$g(x) = x^2 - x$
Find $\left(\frac{f}{g}\right)(x)$

4.
$f(x) = 2x^3 - 5x^2$
$g(x) = 2x - 1$
Find $(f \cdot g)(x)$

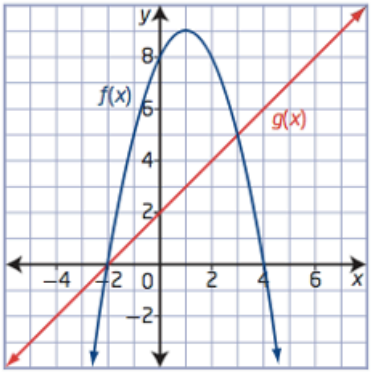
5.
$h(x) = 3x + 3$
$g(x) = -4x + 1$
Find $(h + g)(10)$

6.
$f(x) = 4x - 3$
$g(x) = x^3 + 2x$
Find $(f - g)(4)$

7.
$h(x) = x^2 - 2$
$g(x) = 4x + 1$
Find $(h \circ g)(x)$

8.
$g(n) = 3n + 2$
$f(n) = 2n^2 + 5$
Find $g(f(2))$

9. For each graph of $f(x)$ and $g(x)$, determine the equation and state the domain & range for:
a) $y = (f + g)(x)$ b) $y = \left(\frac{f}{g}\right)(x)$



Answer Key

1. $-t^2 + 2t + 10$

2. $n + 1$

3. $\dfrac{3x - 1}{x^2 - x}$

4. $4x^4 - 12x^3 + 5x^2$

5. -6

6. -59

7. $16x^2 + 8x - 1$

8. 41

9. $a)\ y = -x^2 + 3x + 10\,;\ \ D\!: x \in R,\ \ R\!: y \leq 12.25$

$b)\ y = -x + 4\,;\ \ D\!: x \in R,\ \ R\!: y \in R\,; y \neq -2$

LG 7 Quiz (Function Operations)

1. **Given $f(x) = 4x^2 + 3x - 1$ and $g(x) = 6x + 2$, perform the indicated operation.**

$a)\ (f + g)(x)$ $\qquad$ $b)\ (f - g)(x)$

---

2. **Given $f(x) = 5x - 6$ and $g(x) = x^2 - 5x + 6$, perform the indicated operation.**

$a)\ (g + f)(2)$ $\qquad$ $b)\ (f - g)(-1)$

---

3. **Given $f(x) = 6x^2 - x - 12$ and $g(x) = 2x - 3$, perform the indicated operation.**

$a)\ (fg)(x)$ $\qquad$ $b)\ \left(\frac{f}{g}\right)(x)$

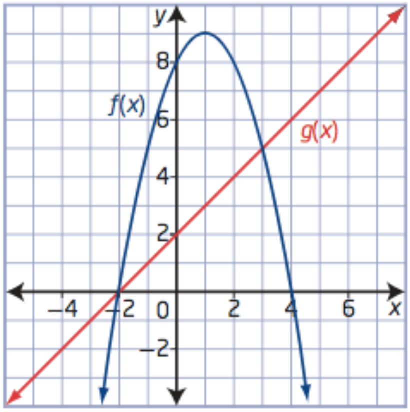
4. **Given $f(x) = 9 - x$, $g(x) = x^2 + x + 1,$ and $h(x) = x - 2$, perform the indicated operation.**

$a)\ (f \circ g)(x)$ $b)\ g(f(x))$ $c)\ h(f(-6))$

---

5. **For each given graph of *f(x)* and *g(x),* perform the indicated operation.**

$a)\ (f + g)(1)$ $b)\ (g - f)(4)$



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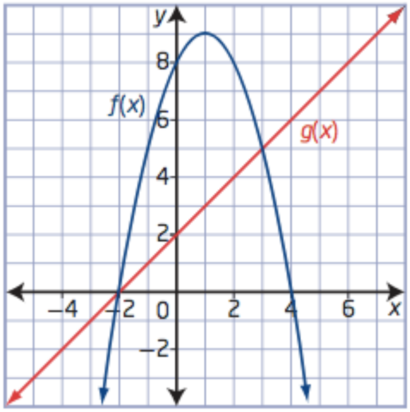
6. **For each graph of *f(x)* and *g(x),* determine the equation and state the domain & range for:**

$a)\ (fg)(x)$ $b)\ \left(\frac{f}{g}\right)(x)$



SEE YOUR TEACHER FOR MARKING KEY

LEARNING GUIDE 8

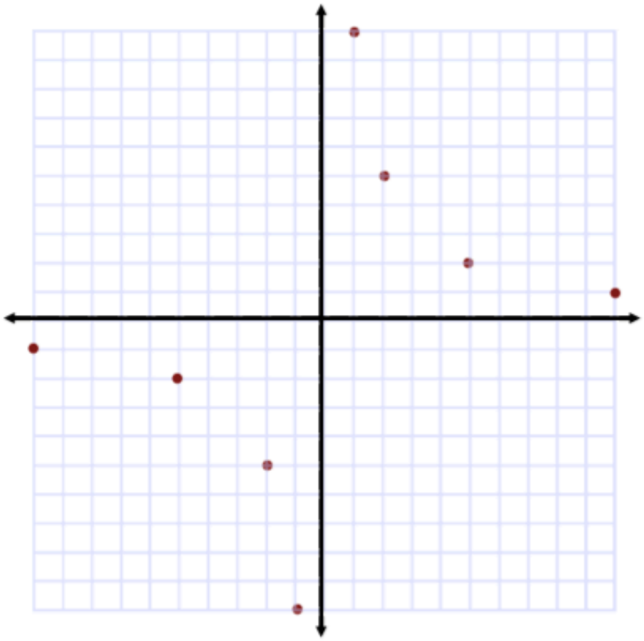
Topic 1 Exploring Rational Functions Using Transformations

EXAMPLE 1 - Graph a Rational Function Using a Table of Values

Analyse the function $y = \frac{10}{x}$ using a table of values and a graph. Identify characteristics of the graph, including the behaviour for the function for its non-permissible.

Solution:

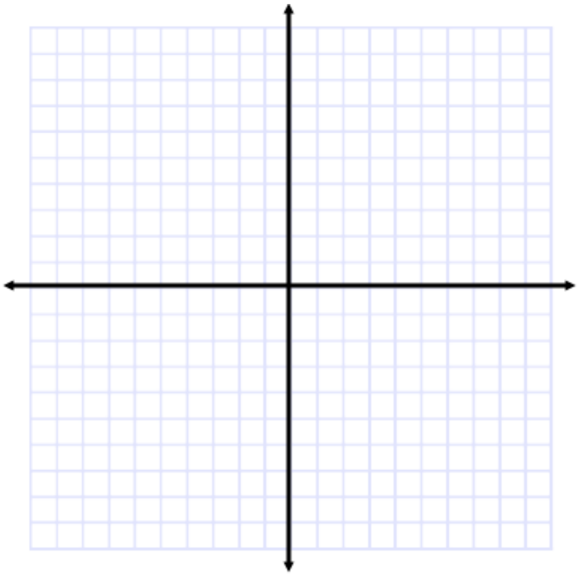
x	y
-100	-0.1
-10	-1
-5	-2
-2	-5
-1	-10
-0.1	-100
0	undefined
0.1	100
1	10
2	5
5	2
10	1
100	0.1



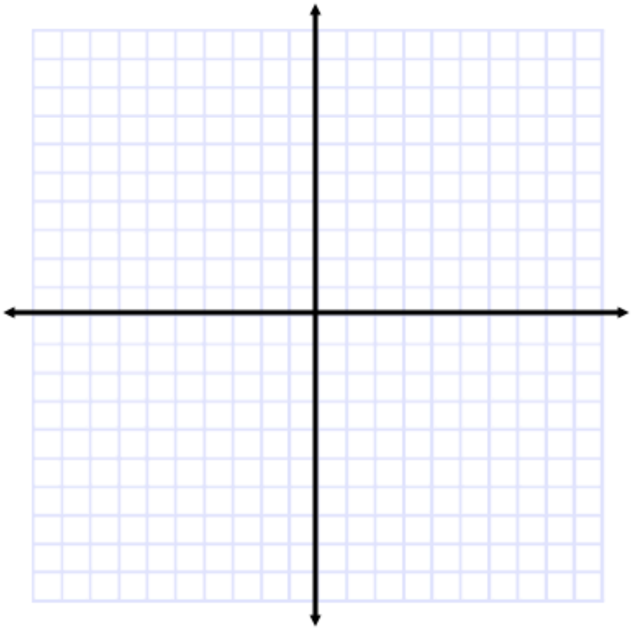
Characteristic	$y = \frac{10}{x}$
Non-permissible value	x = 0
Behaviour near non-permissible value	As x approaches 0, y values become really large or really small.
End behaviour	As y approaches 0, x values become really large or really small.
Domain & Range	{D: x ≠ 0, x ∊ R} & {R: y ≠ 0, y ∊ R}
Equation of vertical asymptote	x = 0
Equation of horizontal asymptote	y = 0

EXAMPLE 2 – Graph Rational Function Using Transformations

Sketch the graph of each function $y = \frac{6}{x-2} - 3$ using transformations, and identify any important characteristics of the graph.

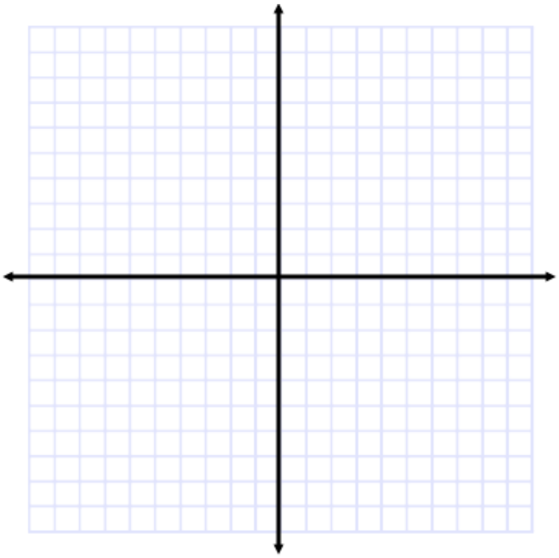


TRY: Sketch the graph of the function $y = \frac{4}{x+1} + 5$ by using transformations. Identify the important characteristics of the graph.



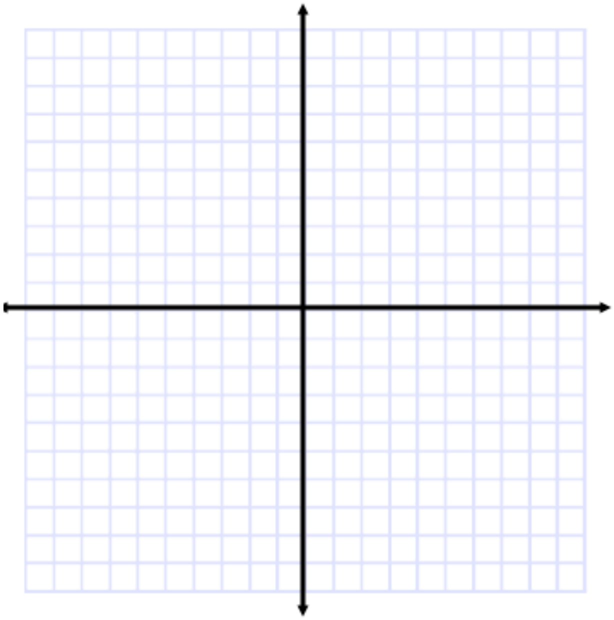
EXAMPLE 3 - Graph a Rational Function With Linear Expressions in Numerator and the Denominator

Graph the function $y = \dfrac{4x-5}{x-2}$. Identify any asymptotes and intercepts.



TRY:

Graph the function $y = \dfrac{2x+2}{x-4}$. Identify any asymptotes and intercepts.



Complete Assignment Questions #1 - #4

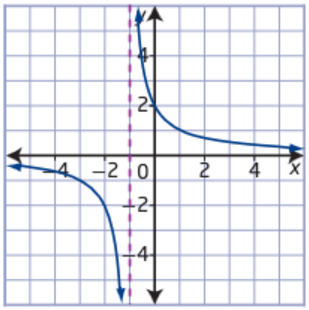
Assignment

1. The equations and graphs of four rational functions are shown. Which graph matches which function? Give reason(s) for each choice.

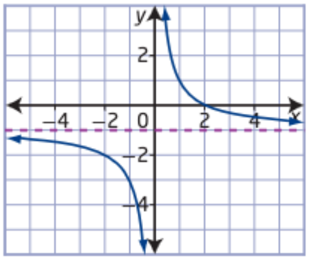
$A(x) = \frac{2}{x} - 1$ $\quad$ $B(x) = \frac{2}{x + 1}$

$C(x) = \frac{2}{x - 1}$ $\quad$ $D(x) = \frac{2}{x} + 1$

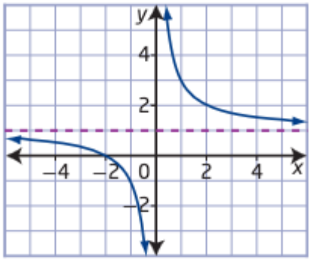
a)



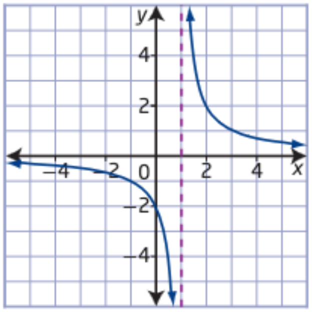
b)



c)



d)

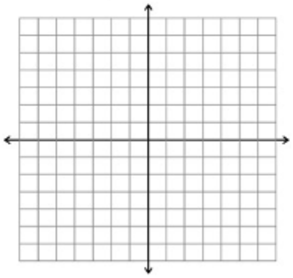


3. Sketch the graph of each function using transformations. Identify the domain and range, intercepts, and asymptotes.

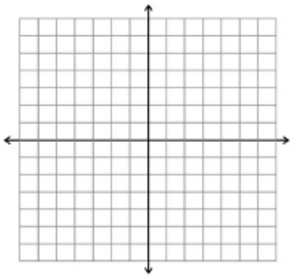
a) $y = \frac{6}{x + 1}$

c) $y = \frac{2}{x - 4} - 5$

a)



c)



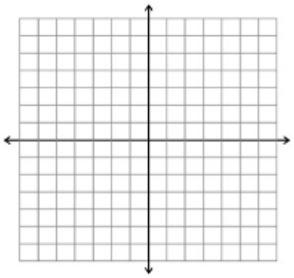
4. Graph each function

identify any asymptotes and intercepts.

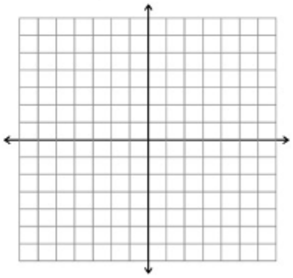
a) $y = \frac{2x + 1}{x - 4}$

c) $y = \frac{-4x + 3}{x + 2}$

a)



c)



Answer Key

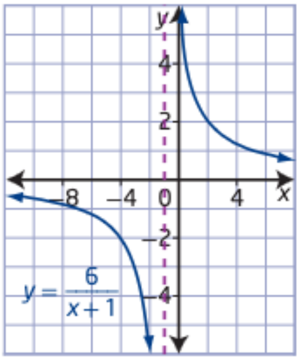
1. a) Since the graph has a vertical asymptote at $x = -1$, it has been translated 1 unit left; $B(x) = \frac{2}{x+1}$.

b) Since the graph has a horizontal asymptote at $y = -1$, it has been translated 1 unit down; $A(x) = \frac{2}{x} - 1$.

c) Since the graph has a horizontal asymptote at $y = 1$, it has been translated 1 unit up; $D(x) = \frac{2}{x} + 1$.

d) Since the graph has a vertical asymptote at $x = 1$, it has been translated 1 unit right; $C(x) = \frac{2}{x-1}$.

3. a)

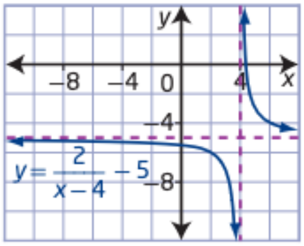


domain $\{x \mid x \neq -1, x \in R\}$, range $\{y \mid y \neq 0, y \in R\}$, no x-intercept, y-intercept 6, horizontal

VA: x = -1

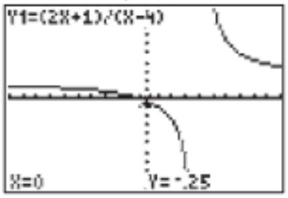
HA: y = 0

b)



domain $\{x \mid x \neq 4, x \in R\}$, range $\{y \mid y \neq -5, y \in R\}$, x-intercept 4.4, y-intercept -5.5, horizontal asymptote $y = -5$,

4. a)

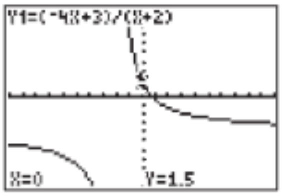


Y1=(2X+1)/(X-4)

X=0 Y=-.25

horizontal asymptote $y = 2$, vertical asymptote $x = 4$, x-intercept -0.5, y-intercept -0.25

c)



Y1=(-4X+3)/(X+2)

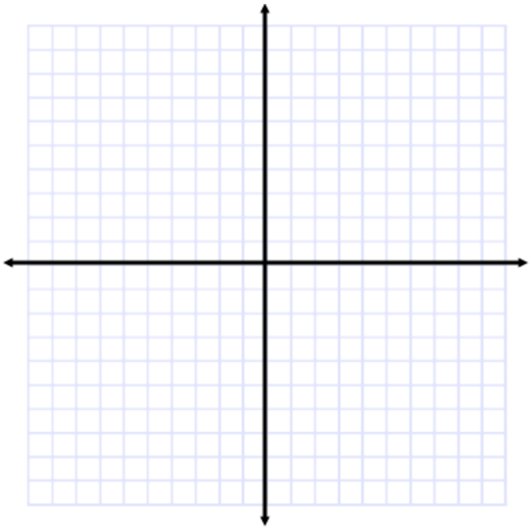
X=0 Y=1.5

horizontal asymptote $y = -4$, vertical asymptote $x = -2$, x-intercept 0.75, y-intercept 1.5

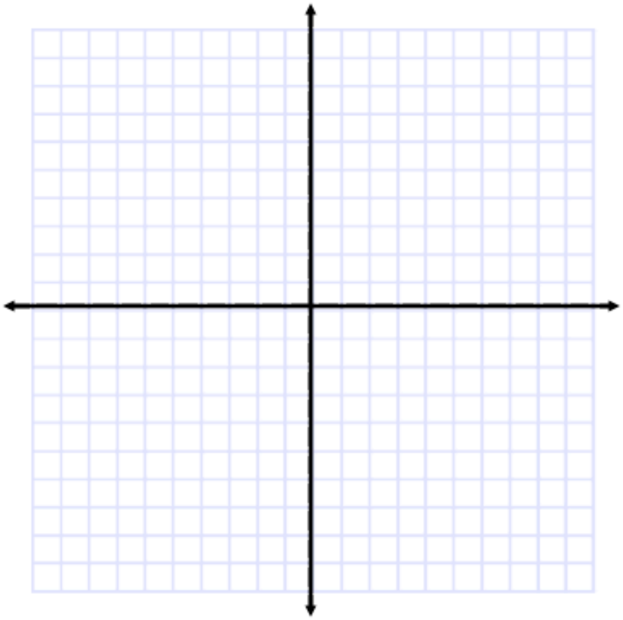
Topic 2 **Analysing Rational Functions**

EXAMPLE 1 - Graph a Rational Function With a Point of Discontinuity

Sketch the graph of the function $y = \frac{x^2 - 5x + 6}{x - 3}$.



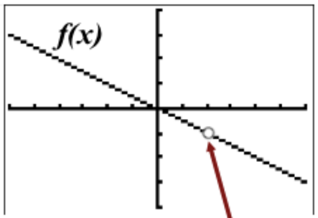
TRY: Sketch the graph of the function $y = \frac{x^2 + 2x - 3}{x - 1}$.

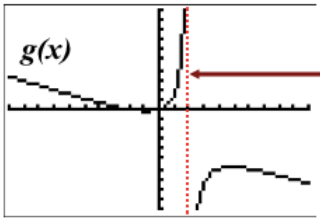


EXAMPLE 2 - Rational Function Point of Discontinuity *versus* Asymptotes

Compare the behaviour of the function $f(x)=\dfrac{x^2-2x}{4-2x}$ and $g(x)=\dfrac{x^2+2x}{4-2x}$.

Solution: ☛ Use your graphing calculator to graph the functions.





How can you tell?

$$f(x)=\frac{x^2-2x}{4-2x}$$

$$f(x)=\frac{x(\cancel{x-2})}{-2(\cancel{x-2})}$$

$$f(x)=-\frac{1}{2}x\ ,\ x\neq 2$$

You can see *f(x)* has a point of discontinuity **(2,-1)**

$$g(x)=\frac{x^2+2x}{4-2x}$$

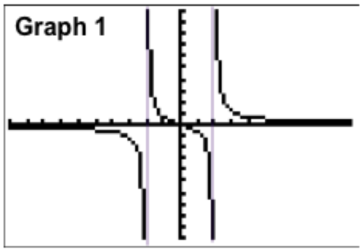
$$g(x)=\frac{x(x+2)}{-2(x-2)}\ ,\ x\neq 2$$

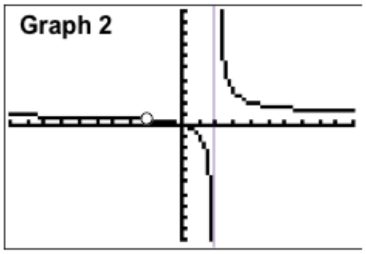
You can see g(x) has a vertical asymptote **$x=2$**

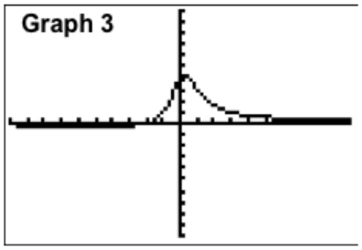
EXAMPLE 3 - Match Graphs and Equations for Rational Functions

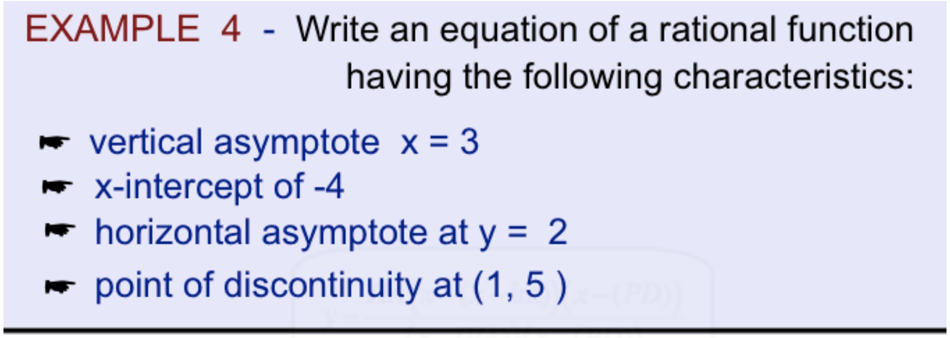
Match the equation of each rational function with the most appropriate graph. Give reasons for choice.

$A(x)=\dfrac{x^2+2x}{x^2-4}$ $\quad B(x)=\dfrac{2x+4}{x^2+1}$ $\quad C(x)=\dfrac{2x}{x^2-4}$


Graph 1

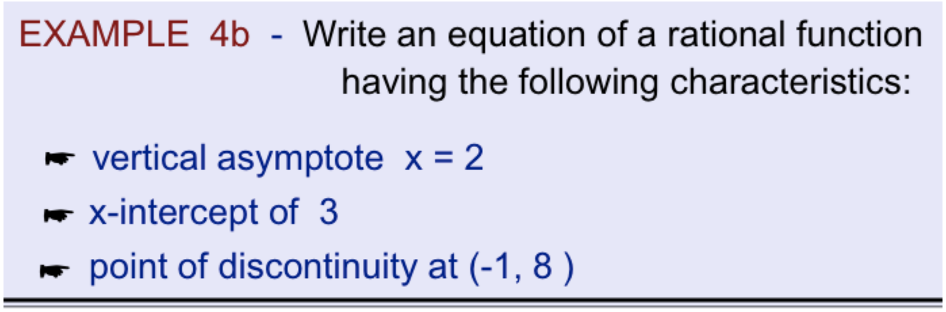

Graph 2


Graph 3



EXAMPLE 4 - Write an equation of a rational function having the following characteristics:

- vertical asymptote x = 3
- x-intercept of -4
- horizontal asymptote at y = 2
- point of discontinuity at (1, 5)



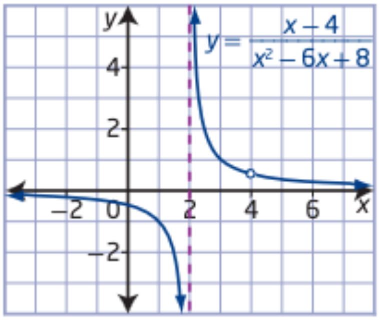
EXAMPLE 4b - Write an equation of a rational function having the following characteristics:

- vertical asymptote x = 2
- x-intercept of 3
- point of discontinuity at (-1, 8)

Complete Assignment Questions #1 - #8

Assignment

1. The graph of the rational function $y = \frac{x - 4}{x^2 - 6x + 8}$ is shown.



a) Copy and complete the table to summarize the characteristics of the function.

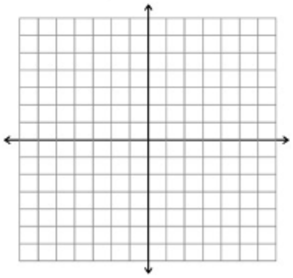
Characteristic	$y = \frac{x - 4}{x^2 - 6x + 8}$
Non-permissible value(s)	
Feature exhibited at each non-permissible value	
Behaviour near each non-permissible value	
Domain	
Range	

4. For each function, predict the locations of any vertical asymptotes, points of discontinuity, and intercepts. Then, graph the function to verify your predictions.

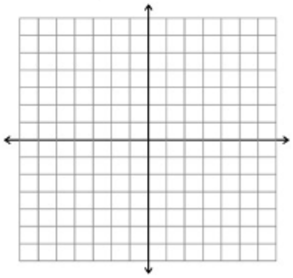
a) $y = \frac{x^2 + 4x}{x^2 + 9x + 20}$

c) $y = \frac{x^2 + 2x - 8}{x^2 - 2x - 8}$

a)



c)



5. Which graph matches each rational function? Explain your choices.

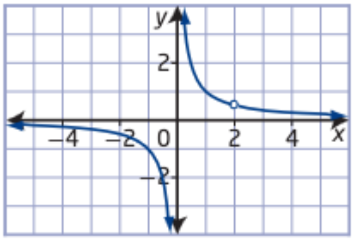
a) $A(x) = \frac{x^2 + 2x}{x^2 + 4}$

b) $B(x) = \frac{x - 2}{x^2 - 2x}$

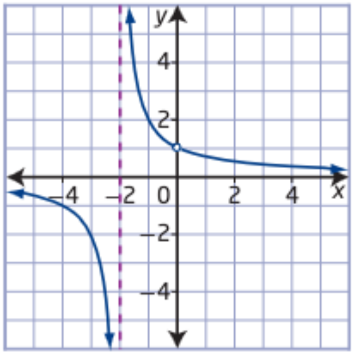
c) $C(x) = \frac{x + 2}{x^2 - 4}$

d) $D(x) = \frac{2x}{x^2 + 2x}$

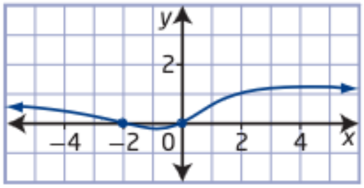
A



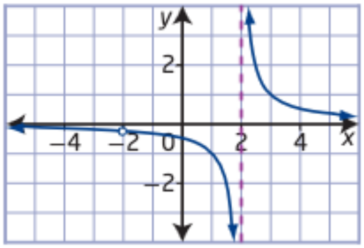
B



C

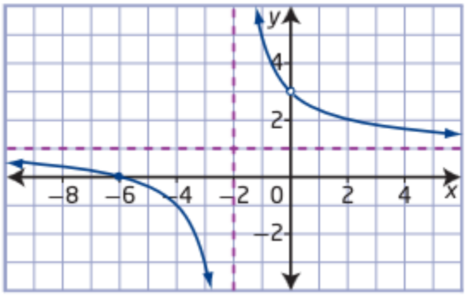


D

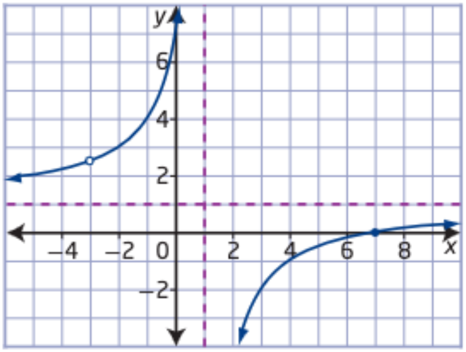


7. Write the equation for each rational function graphed below.

a)



b)



8. Write the equation of a possible rational function with each set of characteristics.

a) vertical asymptotes at $x = \pm 5$ and x-intercepts of -10 and 4

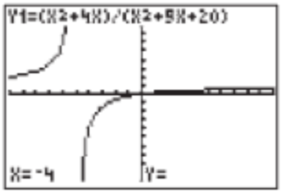
c) a point of discontinuity at $\left(-2, \frac{1}{5}\right)$, a vertical asymptote at $x = 3$, and an x-intercept of -1

Answer Key

1. a)

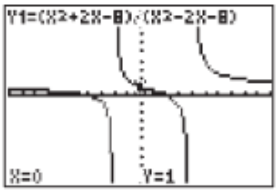
Characteristic	$y = \frac{x-4}{x^2-6x+8}$
Non-permissible value(s)	$x = 2$, $x = 4$
Feature exhibited at each non-permissible value	vertical asymptote, point of discontinuity
Behaviour near each non-permissible value	As x approaches 2, $\lvert y \rvert$ becomes very large. As x approaches 4, y approaches 0.5.
Domain	$\{x \mid x \neq 2, 4, x \in R\}$
Range	$\{y \mid y \neq 0, 0.5, y \in R\}$

4. a)



Vertical asymptote $x = -5$; point of discontinuity $(-4, -4)$; x-intercept 0; y-intercept 0

c)



Vertical asymptotes $x = -2, 4$; no points of discontinuity; x-intercepts $-4, 2$; y-intercept 1

5. a) C b) A c) D d) B

7. a) $y = \frac{x^2 + 6x}{x^2 + 2x}$ **b)** $y = \frac{x^2 - 4x - 21}{x^2 + 2x - 3}$

8. a) $y = \frac{(x + 10)(x - 4)}{(x + 5)(x - 5)}$ **b)** $y = \frac{(2x + 11)(x - 8)}{(x + 4)(2x + 11)}$

c) $y = \frac{(x + 2)(x + 1)}{(x - 3)(x + 2)}$ **d)** $y = \frac{x(4x + 1)}{(x - 3)(7x - 6)}$

Topic 3 Connecting Graphs and Rational Equations

EXAMPLE 1 - Relate Roots and x-intercepts

a) Determine the root(s) of $x+\frac{6}{x+2}-5=0$ algebraically.

b) Using a graph, determine the x-intercept(s) of the graph of $y=x+\frac{6}{x+2}-5$

c) Describe the connection between the root(s) of the equation and the x-intercepts of the graph of the function.

TRY: a) Determine the root(s) of $\frac{14}{x}-x+5=0$ algebraically.

b) Using a graph, determine the x-intercept(s) of the graph of $y=\frac{14}{x}-x+5$.

EXAMPLE 2 - Solve a Rational Equation With an Extraneous Root

Solve the equation $\frac{x}{2x+5}+2x=\frac{8x+15}{4x+10}$ algebraically and graphically.

TRY - Solve a Rational Equation

Solve the equation $\frac{x^2-3x+2}{x-1}=2x+1$ algebraically and graphically.

Complete Assignment Questions #3 - #9

Assignment

3. Solve each equation algebraically.

a) $\frac{5x}{3x+4} = 7$

c) $\frac{x^2}{x-2} = x - 6$

6. Solve each equation algebraically and graphically. Compare the solutions found using each method.

a) $\frac{3x}{x-2} + 5x = \frac{x+4}{x-2}$

c) $\frac{6x}{x-3} + 3x = \frac{2x^2}{x-3} - 5$

8. Determine the solution to the equation $\frac{2x+1}{x-1} = \frac{2}{x+2} - \frac{3}{2}$.

9. Solve the equation $2 - \frac{1}{x+2} = \frac{x}{x+2} + 1$ algebraically.

Answer Key

3. a) $x = -\frac{7}{4}$ **b)** $x = 4$ **c)** $x = \frac{3}{2}$

6. a) $x = -\frac{2}{5}$

c) $x = -5$

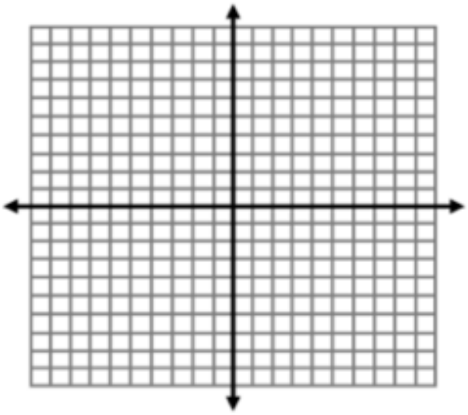
8. $x = -1, x = -\frac{2}{7}$

9. No solutions

LG 8 Worksheet (Rational Functions)

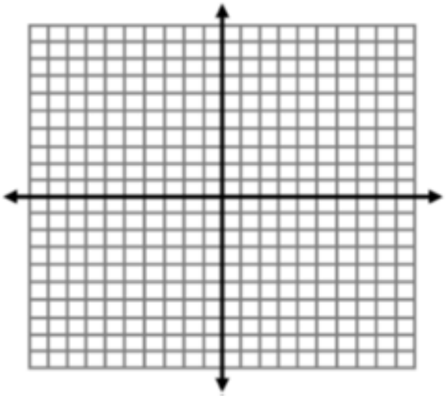
Graph the function and label the following information.

1. $y=\dfrac{1}{x+2}$



x-intercepts:	
Vertical Asymptotes:	
Horizontal Asymptotes:	
Holes: **POD**	
y-Intercept(s):	
Domain:	
Range	

2. $y=\dfrac{x^2+5x+6}{x^2-9}$

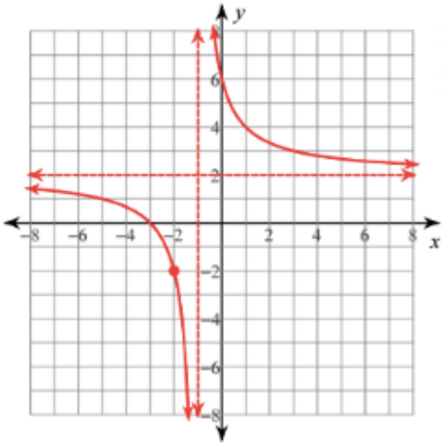


x-intercepts:	
Vertical Asymptotes:	
Horizontal Asymptotes:	
Holes: **POD**	
y-Intercept(s):	
Domain:	
Range	

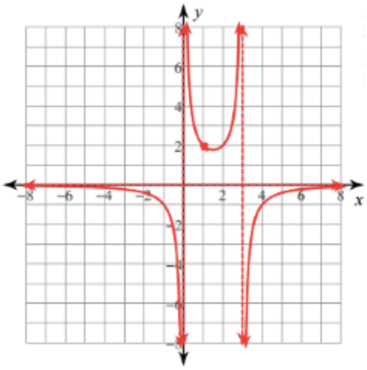
Write a rational function with the given graphs.

Note: points shown on graphs indicate holes.

3.



4.



5. Solve each equation algebraically.

a)

$$\frac{1}{n-8}-1=\frac{7}{n-8}$$

b)

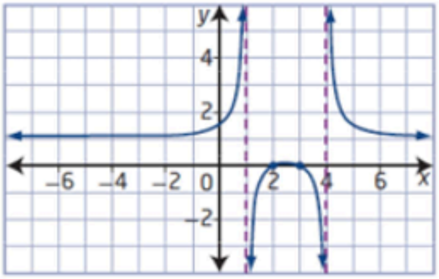
$$1=\frac{1}{x^2+2x}+\frac{x-1}{x}$$

6. Match the following functions to their graph.

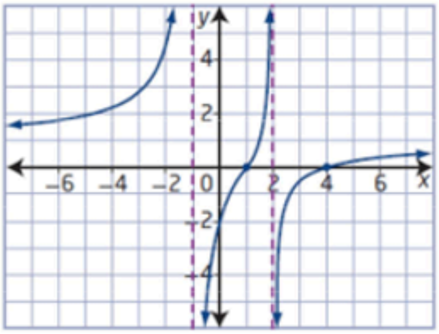
A $f(x) = \frac{x^2+x-2}{x^2+x-20}$ **B** $g(x) = \frac{x^2-5x+4}{x^2-x-2}$

C $g(x) = \frac{x^2-5x+6}{x^2-5x+4}$ **D** $j(x) = \frac{x^2+x-12}{x^2-3x-10}$

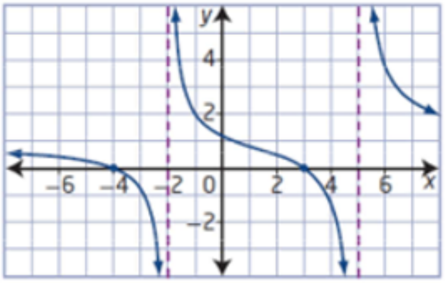
a)



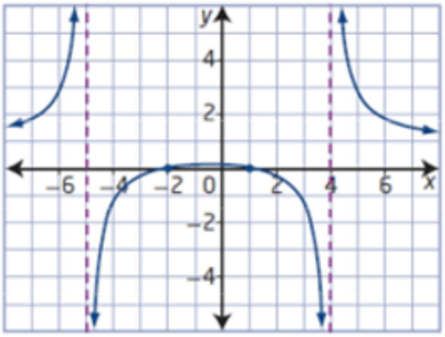
b)



c)

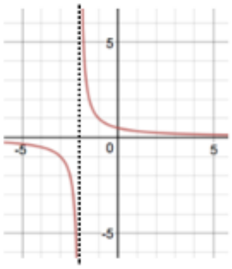


d)



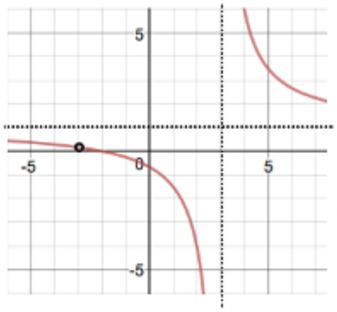
Answer Key

1.



x-intercepts:	none
Vertical Asymptotes:	x = -2
Horizontal Asymptotes:	y = 0
Holes: **POD**	none
y-Intercept(s):	0.5
Domain:	xER x ≠ -2
Range	yER y ≠ 0

2.



x-intercepts:	-2
Vertical Asymptotes:	x = 3
Horizontal Asymptotes:	y = 1
Holes: **POD**	(-3, 1/6)
y-Intercept(s):	-2/3
Domain:	xER x ≠ 3, -3
Range	yER y ≠ 1, 1/6

3. $y = \frac{2(x+3)(x+2)}{(x+1)(x+2)}$ **4.** $y = \frac{-4(x-1)}{x(x-3)(x-1)}$

5 a) 2

b) -1

6. A -> d)

B -> b)

C -> a)

D -> c)

PRE-CALCULUS 12

Seminar Notes

Learning Guides 10 & 11

TRIGONOMETRIC RATIOS

Trigonometric Functions in Degrees

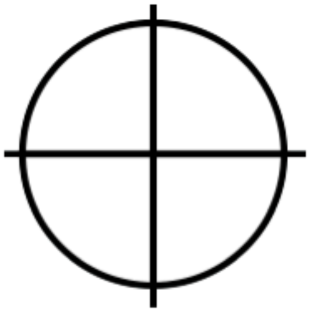
Set Mode to Degrees

Topic 1 Identify Coterminal Angles

EXAMPLE 1 - Draw 30° in standard position.

Terminology:

- *Initial arm*
- *Terminal arm*
- *Quadrant*



Angles that have the same initial and same terminal arm are called coterminal angles.

Give a general way of stating all the angles which are coterminal to 30°.

general solution -

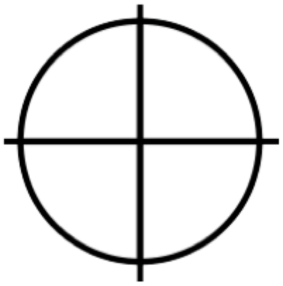
Positive Coterminal

Negative Coterminal

EXAMPLE 1b -

Draw a Negative Angle in Standard Position

Sketch the angle -120˚ in standard position. Draw and give the measure of two coterminal angles. Give a general solution for all coterminal angles to -120˚.



general solution -

If $\theta > 0$, the rotation is **counterclockwise.**

If $\theta < 0$, the rotation is **clockwise.**

This motion is repeated every 360˚; therefore, is called periodic.

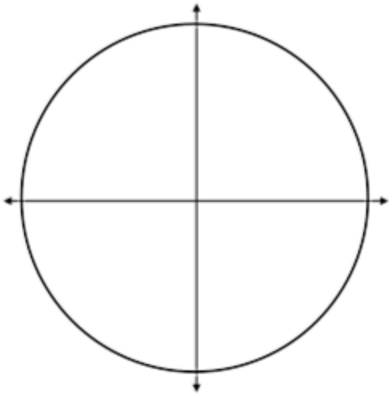
<u>Coterminal Angles</u>

When a position P is described in more than one angle of standard position the angles are said to be coterminal angles.

Coterminal angles can be represented by + 360˚n, in degrees, where *n* is any integer

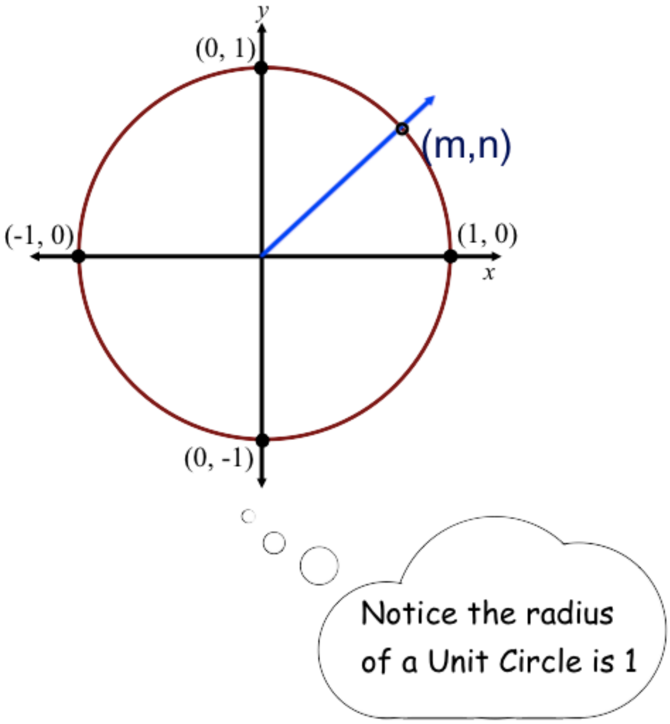
EXAMPLE 2 - Finding Coterminal Angles with unusual Domains

For the angle given, determine all angles that are coterminal in the given domain. $-70°, -720° \le \theta < 360°$



Topic 2 The Unit Circle

EXAMPLE 1 - The point (m,n) is the point of intersection on the unit circle. Find the value of Sin ϴ, Cos ϴ, Tan ϴ?



EXAMPLE 2 - Determine Coordinates for Points of the Unit Circle

Is each point on the unit circle? How do you know?

$a)\ \left(\frac{4}{5}, \frac{-3}{5}\right)$ $b)\ \left(\frac{\sqrt{5}}{8}, \frac{7}{8}\right)$

EXAMPLE 3 - Determine the Missing Coordinate(s) for all points on the Unit Circle

Find the missing coordinate. Draw a digram to support your answer.

$$\left(x, \frac{2}{3}\right)\ in\ quadrant\ II$$

TRY: Find the missing coordinate. Draw a digram to support your answer. $\left(-\frac{5}{8}, y\right)\ in\ quadrant\ III$

Complete Assignment Questions #2 - #13

Assignment

2. Is each point on the unit circle? How do you know?

a) $\left(-\frac{3}{4}, \frac{1}{4}\right)$

c) $\left(-\frac{5}{13}, \frac{12}{13}\right)$

e) $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

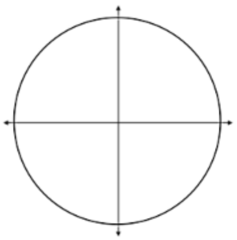
3. Determine the missing coordinate(s) for all points on the unit circle satisfying the given conditions. Draw a diagram to support your answer.

a) $\left(\frac{1}{4}, y\right)$ in quadrant I

c) $\left(-\frac{7}{8}, y\right)$ in quadrant III

e) $\left(x, \frac{1}{3}\right)$, where $x < 0$

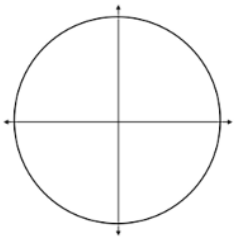
6. Sketch each angle in standard position. In which quadrant does each angle terminate?

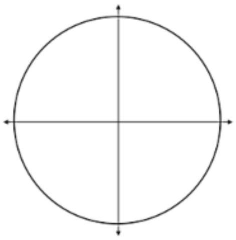


b) $-225°$

d) 650°

f) $-42°$





7. Determine one positive and one negative angle coterminal with each angle.

a) 72°

c) $-120°$

e) $-205°$

8. Determine whether the angles in each pair are coterminal. For one pair of angles, explain how you know.

c) 410°, $-410°$ **d)** 227°, $-493°$

9. Write an expression for all of the angles coterminal with each angle. Indicate what your variable represents.

a) 135°

c) $-200°$

11. For each angle, determine all angles that are coterminal in the given domain.

a) 65°, $0° \leq \theta < 720°$

c) $-40°$, $-720° \leq \theta < 720°$

Answer Key

2. a) No; $\left(-\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{5}{8} \neq 1$

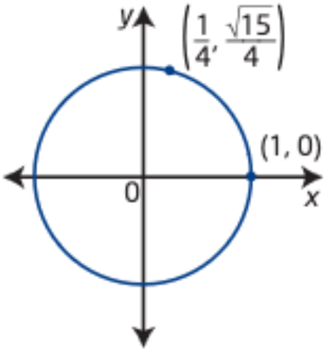
b) No; $\left(\frac{\sqrt{5}}{8}\right)^2 + \left(\frac{7}{8}\right)^2 = \frac{27}{32} \neq 1$

c) Yes; $\left(-\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2 = 1$

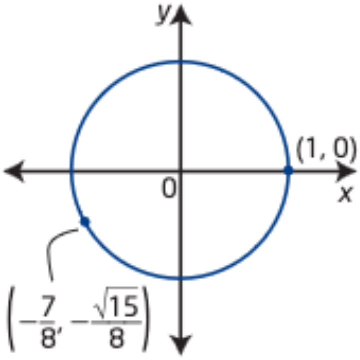
d) Yes; $\left(\frac{4}{5}\right)^2 + \left(-\frac{3}{5}\right)^2 = 1$

e) Yes; $\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 = 1$

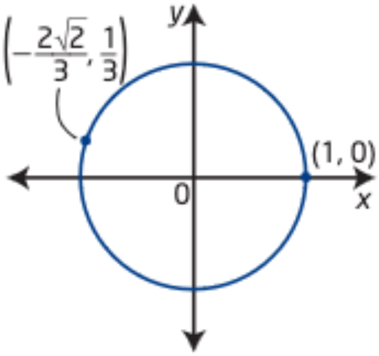
3. a) $y = \frac{\sqrt{15}}{4}$



c) $y = -\frac{\sqrt{15}}{8}$

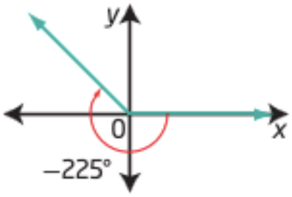


e) $x = -\frac{2\sqrt{2}}{3}$

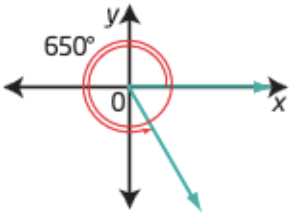


6.

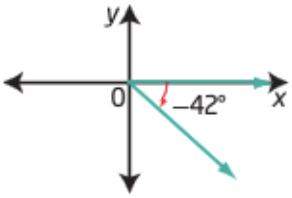
b) quadrant II



d) quadrant IV



f) quadrant IV



7. Examples:

a) 432°, –288°

c) 240°, –480°

e) 155°, –565°

8. a) not coterminal

c) coterminal, -493° = 227° – 2(360°)

9. a) $135° \pm (360°)n,\ n \in \mathrm{N}$

c) $-200° \pm (360°)n,\ n \in \mathrm{N}$

11. a) 425°

c) –400°, 320°, 680°

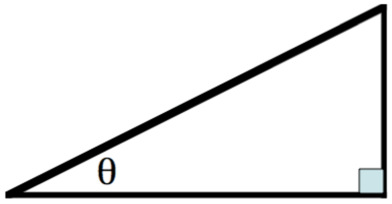
2. Right Triangle Trigonometry

REMEMBER back to Gr. 9

SOH CAH TOA

SOH CAH TOA

$\sin = \frac{opp}{hyp}$ $\cos = \frac{adj}{hyp}$ $\tan = \frac{opp}{adj}$

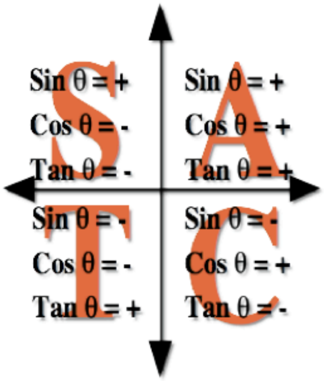


Sin θ **=**

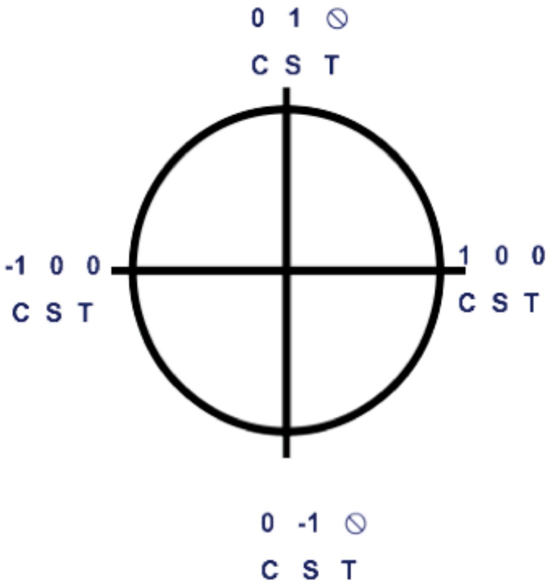
Cos θ **=**

Tan θ **=**

A useful diagram for remembering the sign of the sine, cosine and tangent function of an angle:



Another useful diagram for remembering the sign of the sine, cosine and tangent function of $0^\circ, 90^\circ, 180^\circ,$ $270^\circ, 360^\circ$ angles:



Reciprocal Functions

A. Defining the Reciprocal Functions

Use **SOH CAH TOA** to fill in the following:

- Sin θ =
- Cos θ =
- Tan θ =

- **Csc θ =**
- **Sec θ =**
- **Cot θ =**

Reciprocal Identities:

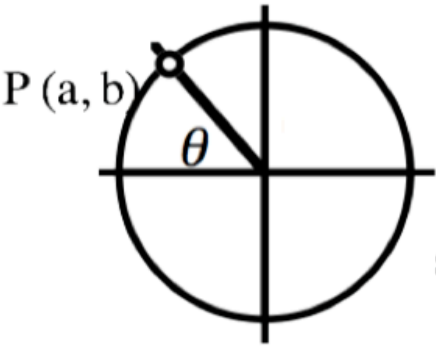
$Csc = \frac{1}{Sin\theta}$ $Sec = \frac{1}{Cos\theta}$ $Cot = \frac{1}{Tan\theta}$

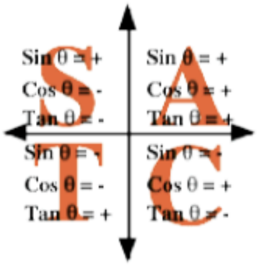
Topic 3

Trigonometric Ratios

EXAMPLE 1 - Given the diagram of a unit circle below *find* $\sin\theta, \cos\theta, \tan\theta, \csc\theta, \sec\theta, \cot\theta.$

Hint: ***Remember to use the CAST rule.***





$\sin\theta =$

$\csc\theta =$

$\cos\theta =$

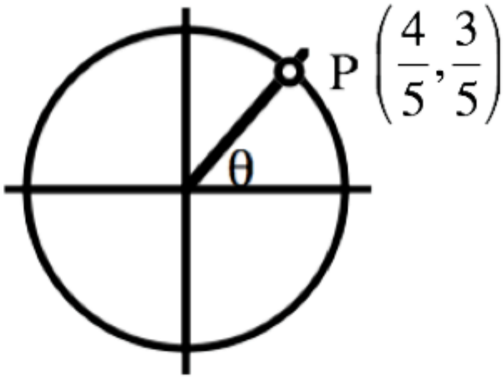
$\sec\theta =$

$\tan\theta =$

$\cot\theta =$

EXAMPLE 2 - Given the diagram below :

find $\sin\theta, \cos\theta, \tan\theta, \csc\theta, \sec\theta, \cot\theta.$



$\sin\theta =$

$\csc\theta =$

$\cos\theta =$

$\sec\theta =$

$\tan\theta =$

$\cot\theta =$

Topic 4

Reference Angles & Exact Values

EXAMPLE 1 - Give the Reference Angle for each the Standard Angles below:

$If\ \theta = 300°;\ R.A. =$

☐

$If\ \theta = 210°;\ R.A. =$

☐

$If\ \theta = 135°;\ R.A. =$

☐

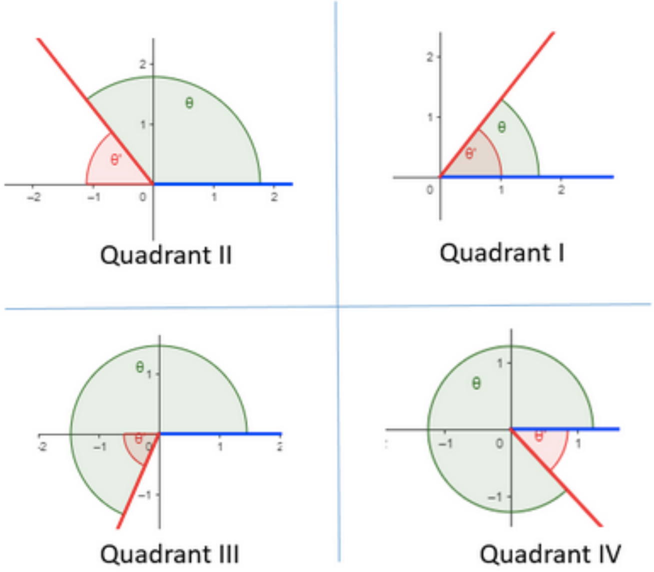
$If\ \theta = 390°;\ R.A. =$

☐

Reference Angle

Standard Angle = θ

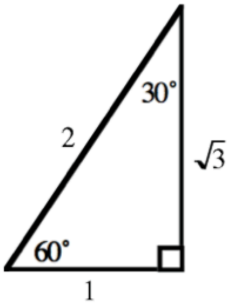
Reference Angle = θ'



Exact Values

Angles 30° & 60° and their multiples

Angles 45° and its multiples



Cos 30° =

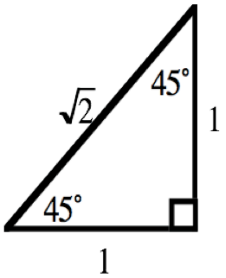
Sin 30° =

Tan 30° =

Cos 60° =

Sin 60° =

Tan 60° =



Cos 45° =

Sin 45° =

Tan 45° =

EXAMPLE 2 - Determine the exact value for:
Cos 135° and Tan 225°

EXAMPLE 2b - Determine exact value for:
a) Cos 120° b) Sin 330°

EXAMPLE 2c - Simplify
$Cos^2(225°)$ *and* $Sin^2(225°)$

✍ Special Angles and Reciprocals

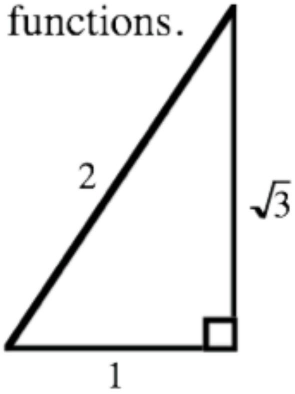
Given $\theta = 60^\circ$, find the six trigonometric functions.

Hint: *draw your special angle diagram.*

Sin θ= Csc θ =

Cos θ= Sec θ =

Tan θ= Cot θ=



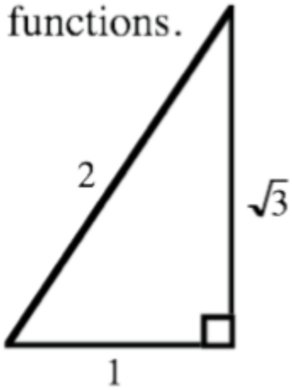
Given $\theta = 30^\circ$, find the six trigonometric functions.

Hint: *draw your special angle diagram.*

Sin θ= Csc θ =

Cos θ= Sec θ =

Tan θ= Cot θ=



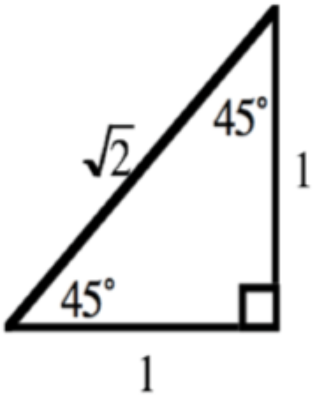
Given $\theta = 45^\circ$, find the six trigonometric functions.

Hint: *draw your special angle diagram.*

Sin θ= Csc θ =

Cos θ= Sec θ =

Tan θ= Cot θ=



EXAMPLE 3 - Determine exact value for:

a) $\mathrm{Csc}^2(120^\circ)$ b) $\mathrm{Cot}^2(240\ )$

Try: - Determine: a) $\mathrm{Sin}^2(225^\circ)$ b) $\mathrm{Sec}^2(300^\circ)$

EXAMPLE 3a -

Given $\theta = 270^\circ$, find the six trigonometric functions.

Hint: ***remember "Unit Circle".***

Sin θ =

Csc θ =

Cos θ =

Sec θ =

Tan θ =

Cot θ =

LG 10 Worksheet A (Ratios and Degrees)

1. Find the exact value of each of the following:

$a)\ \sin 30°$ $b)\ \cos 30°$ $c)\ \sec 45°$ $d)\ \cot 30°$

$e)\ \sin 0°$ $f)\ \tan 0°$ $g)\ \sec 180°$ $h)\ \csc -270°$

2. Find the exact value of each of the following:

$a)\ \sin 150°$ $b)\ \cos 225°$ $c)\ \tan 330°$ $d)\ \cot 135°$

$e)\ \sin -150°$ $f)\ \csc 750°$ $g)\ \sec -300°$ $h)\ \cot 225°$

3. Find the exact value of each of the following:

$a)\ sin^2(60°)$ $b)\ (\sin(60°))^2$ $c)\ sec^2(225°)$ $d)\ tan^3(-30°)$

$e)\ sin^2(90°)$ $f)\ (\sin(-180°))^2$ $g)\ sec^2(270°)$ $h)\ cos^3(540°)$

Answer Key

$1. a)\ \frac{1}{2}$ $b.\ \frac{\sqrt{3}}{2}$ $c. -\sqrt{3}$ $d. -1$ $e.\ 0$ $f.\ 0$ $g. -1$ $h.\ 1$

$2. a)\ \frac{1}{2}$ $b.\ \frac{-1}{\sqrt{2}}$ $c.\ \sqrt{2}$ $d.\ \sqrt{3}$ $e.\ -\frac{1}{2}$ $f.\ 2$ $g.\ 2$ $h.\ 1$

$3. a)\ \frac{3}{4}$ $b.\ \frac{3}{4}$ $c.\ 2$ $d.\ -\frac{1}{3\sqrt{3}}$ $e.\ 1$ $f.\ 0$ $g.\ 0$ $h. -1$

Example 4 - Approximate Values for Trigonometric Ratios

Determine the approximate value for each trig. ratio. Give answers to four decimal places.

a) cos 260° **b) csc (-70°)**

TRY: Find each of the following to 3 dec. places:

$a)\ \sin 160°$ $b)\ \csc -60°$ $c)\ \sec 200°$ $d)\ \cot 305°$

Example 4a - Find Angles Given Their Trigonometric Ratios

Determine the measures of all angles that satisfy the following. Use a diagram in your explanation.

$a)\ \sin\theta = 0.879\ \textit{in the domain}\ 0° \le \theta < 360°.$

Give answers to nearest tenth of a degree.

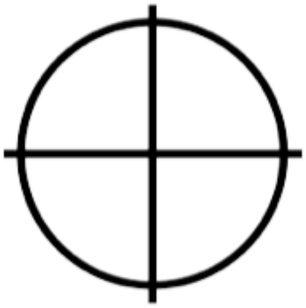
$b)\ \sec\theta = 1.245\ \textit{in the domain}\ -180° \le \theta < 180°.$

Give answers to nearest tenth of a degree.

Example 5 - Find Ratios Given a Point

a) Find the six trigonometric functions, given that P(3, 5) is a point on the terminal of an angle in standard position.

Hint: *use Pythagoras to find the hypotenuse.*



Sin θ = Cscθ =

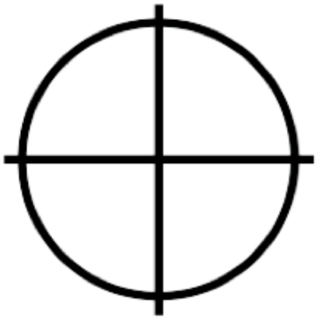
Cos θ = Sec θ =

Tanθ = Cotθ =

Try:

Find the six trigonometric functions, given that P(12, -5) is a point on the terminal of an angle in standard position.

Hint: *use Pythagoras to find the hypotenuse.*



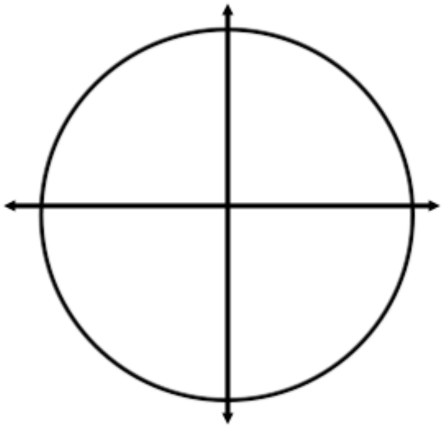
Sin θ = Csc θ =

Cos θ = Sec θ =

Tanθ = Cot θ =

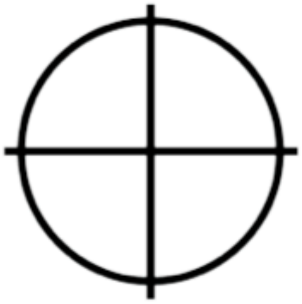
Example 5b - **Finding Ratios given a Ratio**

$$Given\ \ Sin\theta = \frac{4}{5}\ ,\ \ find\ \ Cos\theta\ \ and\ \ Tan\theta.$$



EXAMPLE 5c - a) Find the other five trigonometric functions, given $\text{Cos}\ \theta = \frac{5}{13}$ and θ is an angle in Quadrant IV.

Hint: *use Pythagoras to find the third side.*



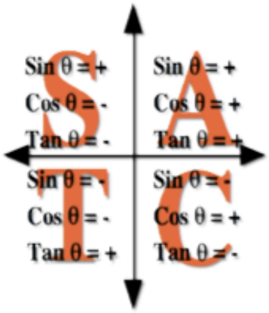
$\text{Sin}\ \theta =$ $\text{Csc}\ \theta =$

$\text{Cos}\ \theta =$ $\text{Sec}\ \theta =$

$\text{Tan}\ \theta =$ $\text{Cot}\ \theta =$

Double Checklist to Locate Quadrant

Examples



1. Sin < 0 and Tan > 0

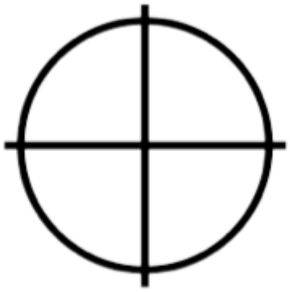
2. Cos < 0 and Cot < 0

3. Sin > 0 and $-90° \le A < 90°$

EXAMPLE 5d

Find the other five trigonometric functions, given $\text{Cos } \theta = \frac{-5}{13}$ and $\text{Sin } \theta > 0$.

Hint: *use Pythagoras to find the third side.*



$\text{Sin } \theta =$

$\text{Csc } \theta =$

$\text{Cos } \theta =$

$\text{Sec } \theta =$

$\text{Tan } \theta =$

$\text{Cot } \theta =$

TRY: If $SinA = \frac{1}{\sqrt{2}}$, find all possible values of SecA.

TRY: If $SecA = -2 \;\; and \;\; 0 < A < 360°$, find all possible values of *CotA* exactly.

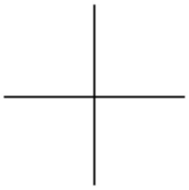
TRY: If $CosA = \frac{-1}{5} \;\; and \;\; SinA < 0$,

find all possible values of *CscA*.

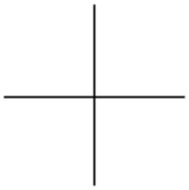
LG 10 Worksheet B (Ratios from Ratios)

If $\angle B$ is an angle in standard position, in which quadrants may $\angle B$ terminate if:

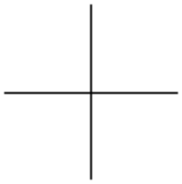
1. $\csc B < 0$



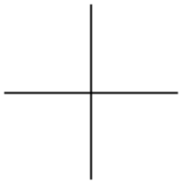
2. $\cot B > 0$



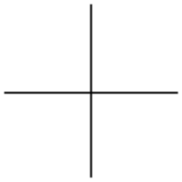
3. $\sin B < 0$ and $180^\circ \leq B < 360^\circ$



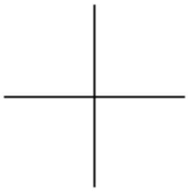
4. $\tan B > 0$ and $90^\circ \leq B < 270^\circ$



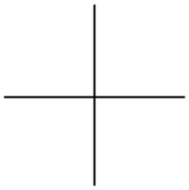
5. $\sec B < 0$ and $180^\circ \leq B < 360^\circ$



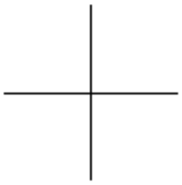
6. $\sin B > 0$ and $\tan B < 0$



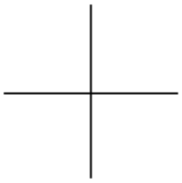
7. $\csc B < 0$ and $\cot B < 0$



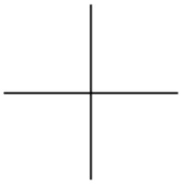
8. $\sec B < 0$ and $\tan B > 0$



9. $\sin B < 0$ and $-90^\circ \leq B < 90^\circ$

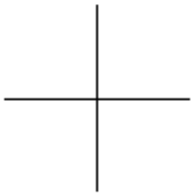


10. $\csc B > 0$ and $-180^\circ \leq B < 180^\circ$

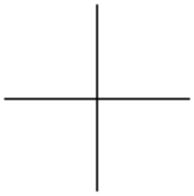


If $cos A = \frac{-3}{5}$, find all of the values of $cot A$ when:

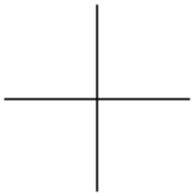
11. $0^\circ \leq A < 360^\circ$



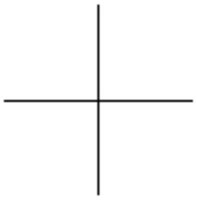
12. $-180^\circ \leq A < 90^\circ$



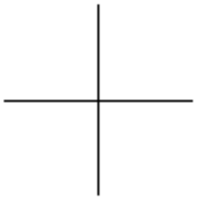
13. $\csc A > 0$



14. $\angle A$ terminates in quadrant III



15. $-90^\circ \leq A < 90^\circ$



16. The point (p, q) is a point of intersection of the terminal arm of $\angle\theta$ in standard position and the unit circle centered at (0, 0). What is the value of $csc\theta$?

Answer Key

1.	III, IV
2.	I, III
3.	III, IV
4.	III
5.	III
6.	II
7.	IV
8.	III
9.	IV
10.	I, II
11.	$\frac{3}{4}, -\frac{3}{4}$
12.	$\frac{3}{4}$
13.	$-\frac{3}{4}$
14.	$\frac{3}{4}$
15.	No soln.
16.	$\frac{1}{q}$

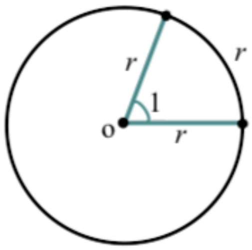
LEARNING GUIDE 11

Trigonometric Functions in Radians

Set Mode to
Radians

Radian

- ☛ one radian is the measure of the central angle that where the arc length is equal to the radius of that circle.



☛ $\pi = 180^\circ \Rightarrow \textit{half rotation}$

☛ $2\pi = 360^\circ \Rightarrow 1\ \textit{full rotation}$

Important Angles :

Degree	Radian
180°	
360°	
90°	
270°	
60°	
30°	
45°	

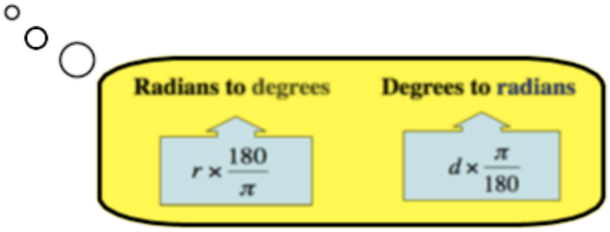
Degree	Radian
120°	
150°	
210°	
300°	
135°	
225°	
315°	

Topic 1

Conversion Factors

EXAMPLE 1 - **Convert each degree measure to radians and each radian measure to degree. Give answers in exact and approximate to the nearest hundredth**

$a)\ 40°$ $b)\ -120°$ $c)\ \frac{2\pi}{3}$ $d)\ -2.5$



TRY: **Convert each degree measure to radians and each radian measure to degree. Give answers in exact and approximate to the nearest hundredth.**

$a)\ 230°$ $b)\ \frac{4\pi}{5}$

EXAMPLE 2 - Coterminal Angles

Determine one positive coterminal and one negative coterminal angle for $\frac{5\pi}{6}$.

TRY: Determine one positive coterminal and one negative coterminal angle: $a)\ \frac{4\pi}{3}$ $b)\ -\frac{3\pi}{4}$

Example 3 - Express Coterminal Angles in General Form

a) Express the angles coterminal with $\frac{2\pi}{3}$ in general form.

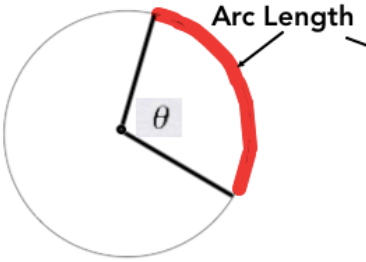
b) Identify angles in the domain $-2\pi \leq \theta < 4\pi$.

Arc Length

- **As a result of the definition of a radian, a relationship between the arc length of a circle and the angle subtended by the radii can be determined:**

$$\frac{angle\ \ at\ \ center}{2\pi} = \frac{arc\ \ length}{circumference}$$

$$\frac{\theta}{2\pi} = \frac{a}{2\pi r}$$

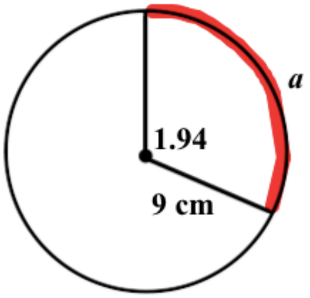


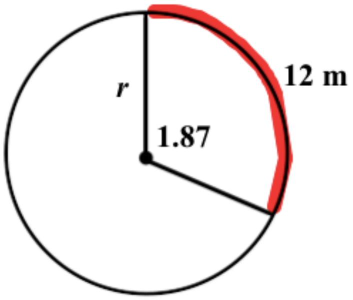
$$a = r\theta$$

$$\boxed{a = r\theta}$$

Example 4 - Determine the Arc Length in a Circle

Use the information in each diagram to determine the value of the variable.





Complete Assignment Questions #6 - #13

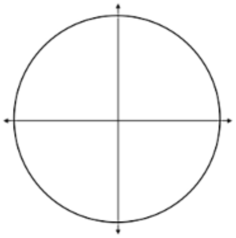
Assignment

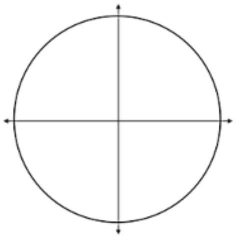
6. Sketch each angle in standard position. In which quadrant does each angle terminate?

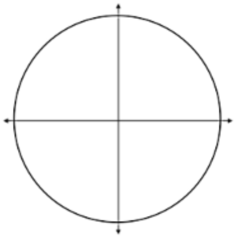
a) 1

c) $\frac{17\pi}{6}$

e) $-\frac{2\pi}{3}$







7. Determine one positive and one negative angle coterminal with each angle.

b) $\frac{3\pi}{4}$

d) $\frac{11\pi}{2}$

f) 7.8

8. Determine whether the angles in each pair are coterminal. For one pair of angles, explain how you know.

a) $\frac{5\pi}{6}, \frac{17\pi}{6}$

b) $\frac{5\pi}{2}, -\frac{9\pi}{2}$

9. Write an expression for all of the angles coterminal with each angle. Indicate what your variable represents.

b) $-\frac{\pi}{2}$

d) 10

11. For each angle, determine all angles that are coterminal in the given domain.

d) $\frac{3\pi}{4}, -2\pi \leq \theta < 2\pi$

f) $\frac{7\pi}{3}, -2\pi \leq \theta < 4\pi$

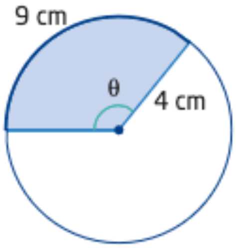
12. Determine the arc length subtended by each central angle. Give answers to the nearest hundredth of a unit.

a) radius 9.5 cm, central angle 1.4

c) radius 7 cm, central angle 130°

13. Use the information in each diagram to determine the value of the variable. Give your answers to the nearest hundredth of a unit.

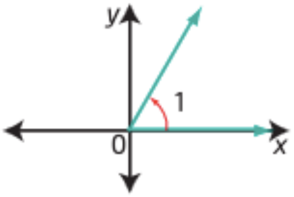
a)



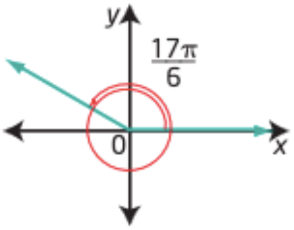
b)

Answer Key

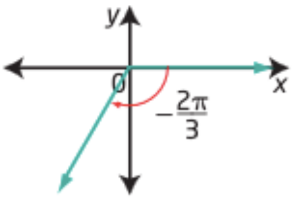
6. a) quadrant I



c) quadrant II



e) quadrant III



7.

b) $\frac{11\pi}{4}, -\frac{5\pi}{4}$

d) $\frac{7\pi}{2}, -\frac{\pi}{2}$

f) 1.5, −4.8

8. a) coterminal, $\frac{17\pi}{6} = \frac{5\pi}{6} + \frac{12\pi}{6} = \frac{5\pi}{6} + 2\pi$

b) not coterminal

9.

b) $-\frac{\pi}{2} \pm 2\pi n, n \in \mathrm{N}$

d) $10 \pm 2\pi n, n \in \mathrm{N}$

11.

d) $-\frac{5\pi}{4}$

f) $-\frac{5\pi}{3}, \frac{\pi}{3}$

12. a) 13.30 cm

c) 15.88 cm

13. a) 2.25 radians

b) 10.98 ft

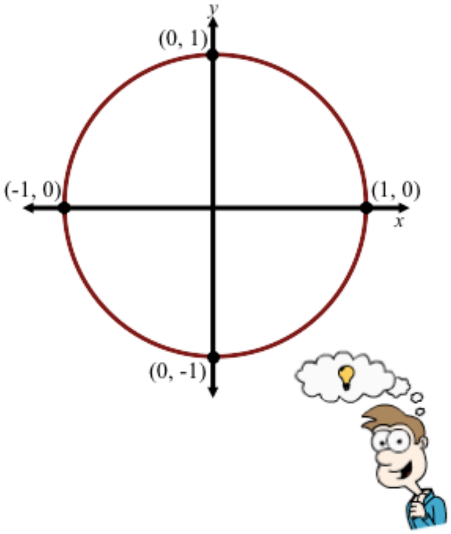
Topic 2

The Unit Circle

Example 1

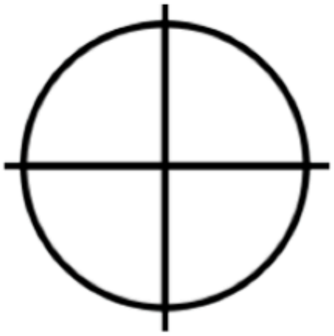
Determine the exact coordinate of each.

$a)\ 3\pi$ $\qquad$ $b)\ -\frac{3\pi}{2}$



EXAMPLE 1b -

Draw the unit circle and evaluate the following:

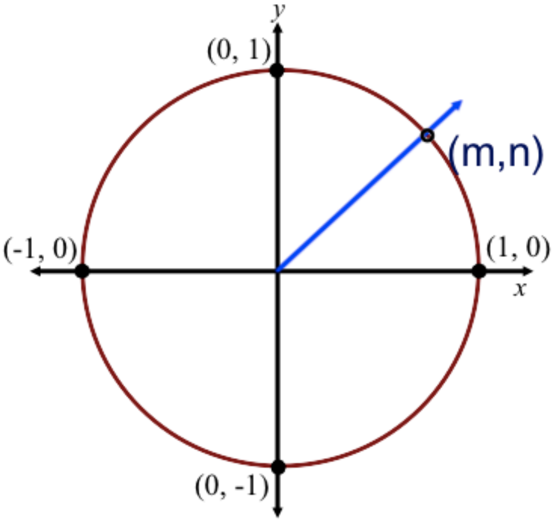


$\text{Cos } 0 =$	$\text{Sin } 0 =$
$\text{Cos } \frac{\pi}{2} =$	$\text{Sin } \frac{\pi}{2} =$
$\text{Cos } \pi =$	$\text{Sin } \pi =$
$\text{Cos } \frac{3\pi}{2} =$	$\text{Sin } \frac{3\pi}{2} =$
$\text{Cos } 2\pi =$	$\text{Sin } 2\pi =$

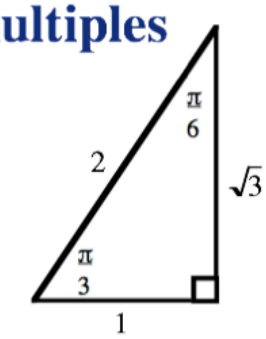
Note: ***as P rotates around the unit circle the values of sine and cosine are always between −1 and 1.***

The Unit Circle

EXAMPLE 1 - The point (m,n) is the point of intersection on the unit circle. Find the value of Sinθ, Cosθ, Tanθ?



Angles $\frac{\pi}{6}$ & $\frac{\pi}{3}$ and their multiples



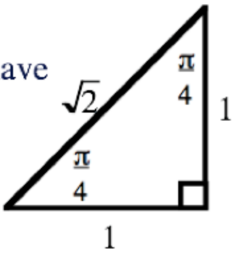
Find the exacts of following :

$\text{Cos}\frac{\pi}{6} =$ $\text{Sin}\frac{\pi}{6} =$ $\text{Tan}\frac{\pi}{6} =$

$\text{Cos}\frac{\pi}{3} =$ $\text{Sin}\frac{\pi}{3} =$ $\text{Tan}\frac{\pi}{3} =$

Angle $\frac{\pi}{4}$ and its multiples

For a $\frac{\pi}{4}$ - $\frac{\pi}{4}$ triangle the following angles have *exact values*



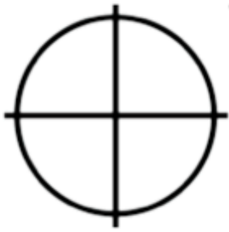
$\text{Cos}\frac{\pi}{4} =$ $\text{Sin}\frac{\pi}{4} =$ $\text{Tan}\frac{\pi}{4} =$

Topic 3 Special Angles

EXAMPLE 1 - Given the angle in radians, find its reference angle

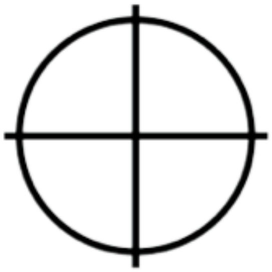
☛1 **Find the Reference Angle for** $\frac{2\pi}{3}$

Draw at right $\frac{2\pi}{3}$.



☛2 **Find the Reference Angle for** $\frac{5\pi}{6}$

Draw at right $\frac{5\pi}{6}$.



EXAMPLE 1b -

Find the exact value for $\text{Cos}\,\frac{2\pi}{3}$ **and** $\text{Sin}\,\frac{7\pi}{6}$

Hint: ***find the reference angle by drawing the angle in standard position first.***

TRY:

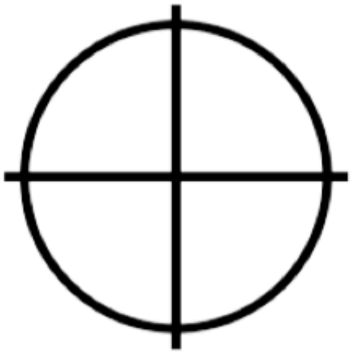
Find the exact $\text{Cos}\,\frac{5\pi}{3}$ **and** $\text{Sin}\,\frac{5\pi}{6}$

Find the exact $\text{Cos}\,\frac{-\pi}{3}$ **and** $\text{Sin}\,\frac{-\pi}{6}$

EXAMPLE 1c - Given the angle in radians, find its reference angle

Find the Reference Angle for $\frac{3\pi}{4}$

Draw at right $\frac{3\pi}{4}$ **.**

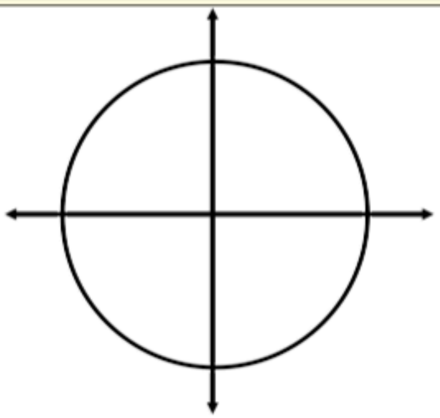


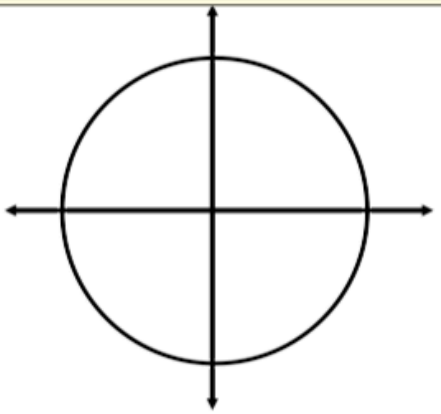
EXAMPLE 1d -

Find the exact value for Cos $\frac{5\pi}{4}$ **and** Sin $\frac{5\pi}{4}$

TRY: **Determine the exact value for each. Draw diagrams to illustrate your answers.**

$a)\ \tan -\frac{5\pi}{3}$ $b)\ \cos \frac{11\pi}{6}$





6. Radians & Reciprocal Functions

Defining the Reciprocal Functions

Use SOH CAH TOA to fill in the following:

Sin θ =

Cos θ =

Tan θ =

Reciprocal Identities:

$Sec\theta = \frac{1}{Cos\theta}$ $Csc\theta = \frac{1}{Sin\theta}$ $Cot\theta = \frac{1}{Tan\theta}$

$Sec\theta =$ $Csc\theta =$ $Cot\theta =$

EXAMPLE 2 - Special Angles & Reciprocals

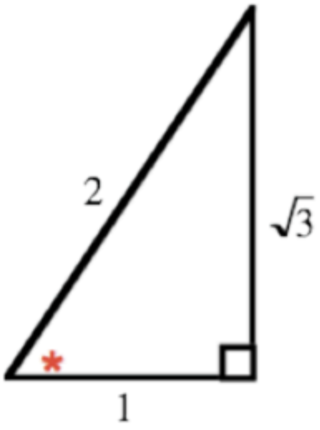
Given $\theta = \frac{\pi}{3}$, find the six trigonometric functions.

Hint: ***draw your special angle diagram.***

$Sin\theta =$ $Csc\theta =$

$Cos\theta =$ $Sec\theta =$

$Tan\theta =$ $Cot\theta =$



EXAMPLE 2b - More Special Angles & their Reciprocals

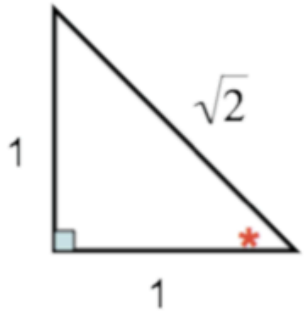
Given $\theta = \frac{3\pi}{4}$, find the six trigonometric functions.

Hint: *draw your special angle diagram.*

$Sin\theta =$ $Csc\theta =$

$Cos\theta =$ $Sec\theta =$

$Tan\theta =$ $Cot\theta =$



EXAMPLE 3 - More Special Angles & their Reciprocals

Determine the exact value for:

$a)\ Csc^2 \frac{5\pi}{4}$ $b)\ Sec^3 \frac{7\pi}{6}$

Hint: ***find the reference angle by drawing the angle in standard position first, then use the CAST rule to determine the sign of the function.***

LG 11 Worksheet A (Ratios and Radians)

1. Find the exact value of each of the following:

$a)\ \sin(0)$ $b)\ \cos(\pi)$ $c)\ \tan\left(\frac{\pi}{2}\right)$ $d)\ \sec\left(\frac{-3\pi}{2}\right)$

$e)\ \csc\left(\frac{-3\pi}{2}\right)$ $f)\ \cot(\pi)$ $g)\ \sin\left(\frac{9\pi}{2}\right)$ $h)\ \cos(11\pi)$

2. Find the exact value of each of the following:

$a)\ \sin\left(\frac{5\pi}{6}\right)$ $b)\ \cos\left(\frac{5\pi}{4}\right)$ $c)\ \tan\left(\frac{5\pi}{3}\right)$ $d)\ \csc\left(\frac{4\pi}{3}\right)$

$e)\ \sec\left(\frac{11\pi}{6}\right)$ $f)\ \cot\left(\frac{3\pi}{4}\right)$ $g)\ \sin\left(\frac{-5\pi}{6}\right)$ $h)\ \sec\left(\frac{-5\pi}{3}\right)$

3. Find the exact value of each of the following:

$a)\ sin^2\left(\frac{\pi}{3}\right)$ $b)\ \left(cos\left(\frac{2\pi}{3}\right)\right)^2$ $c)\ sec^2\left(\frac{5\pi}{4}\right)$ $d)\ tan^2\left(\frac{-\pi}{6}\right)$

$e)\ csc^3\left(\frac{7\pi}{6}\right)$ $f)\ cot^4\left(\frac{7\pi}{4}\right)$ $g)\ cos^3\left(\frac{-5\pi}{4}\right)$ $h)\ sec^2(3\pi)$

Answer Key

1. a) 0 b) -1 c) undefined d) undef e) 1 f) undef. g) 1 h) -1

2. a) $\frac{1}{2}$ $b)\ \frac{-1}{\sqrt{2}}$ $c)\ -\sqrt{3}$ $d)\ \frac{-2}{\sqrt{3}}$ $e)\ \frac{2}{\sqrt{3}}$ $f)\ -1$ $g)\ \frac{-1}{2}$ $h)\ 2$

3. a) $\frac{3}{4}$ $b)\ \frac{1}{4}$ $c)\ 2$ $d)\ \frac{1}{3}$ $e)\ -8$ $f)\ 1$ $g)\ \frac{-1}{2\sqrt{2}}$ $h)\ 1$

Topic 3 Approximate Values for Trig. Ratios

EXAMPLE 1 - Find the value to 3 decimal places

$a)\ Sin\frac{3\pi}{5}$ $b)\ Cos\frac{5\pi}{7}$

$c)\ Tan\frac{-\pi}{8}$ $d)\ Csc\frac{2\pi}{5}$

$e)\ Sec\frac{11\pi}{7}$

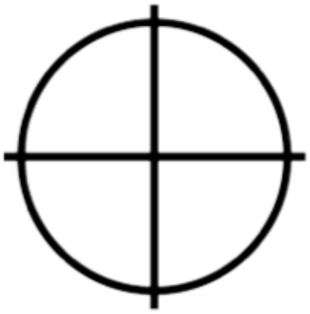
TRY: Find each of the following to 3 dec. places:

$a)\ \sin\left(\frac{5\pi}{7}\right)$ $b)\ \cot\left(\frac{11\pi}{5}\right)$ $c)\ \tan 2.5$ $d)\ \csc\left(\frac{3}{7\pi}\right)$

Example 2

a) Find the other five trigonometric functions, given
Cos $\theta = \frac{5}{13}$ and θ is an angle in Quadrant IV.

Hint: *use Pythagoras to find the third side.*



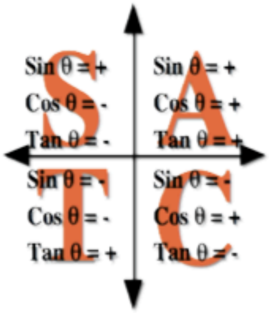
Sin $\theta =$ Csc $\theta =$

Cos $\theta =$ Sec $\theta =$

Tan $\theta =$ Cot $\theta =$

Double Checklist to Locate Quadrant

Examples



1. Sin < 0 and Tan > 0

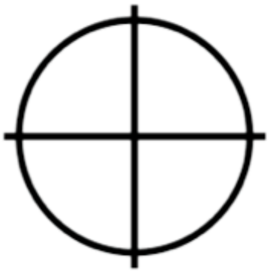
2. Cos < 0 and Cot < 0

3. Sin > 0 and $\frac{-\pi}{2} \leq A < \frac{\pi}{2}$

Example 2

b) Find the other five trigonometric functions, given $\text{Cos}\ \theta = \frac{-5}{13}$ and $\text{Sin}\ \theta > 0$.

Hint: *use Pythagoras to find the third side.*

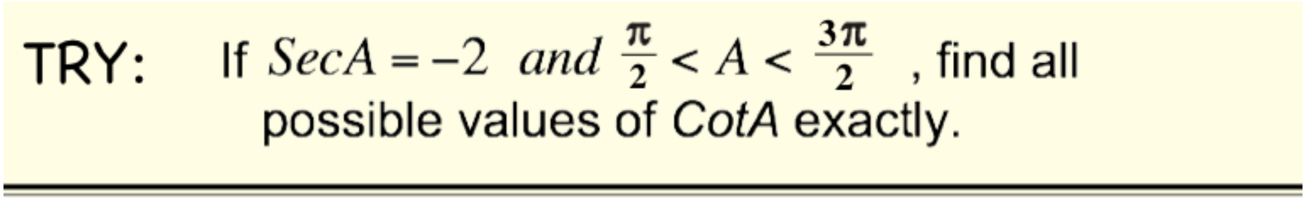


$\text{Sin}\ \theta =$	$\text{Csc}\ \theta =$
$\text{Cos}\ \theta =$	$\text{Sec}\ \theta =$
$\text{Tan}\ \theta =$	$\text{Cot}\ \theta =$

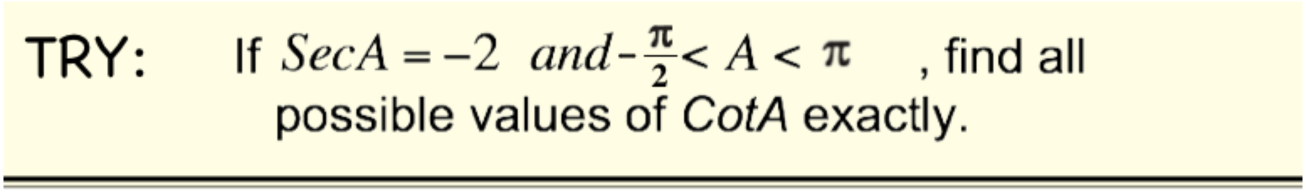
Example 3

If $SinA = \frac{1}{\sqrt{2}}$, find all possible values of SecA.

TRY: If $SecA = -2\ \ and\ \ 0 < A < 2\pi$, find all possible values of *CotA* exactly.



TRY: If $SecA = -2 \;\; and \; \frac{\pi}{2} < A < \frac{3\pi}{2}$, find all possible values of *CotA* exactly.

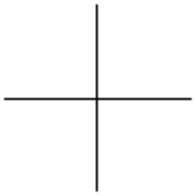


TRY: If $SecA = -2 \;\; and - \frac{\pi}{2} < A < \pi$, find all possible values of *CotA* exactly.

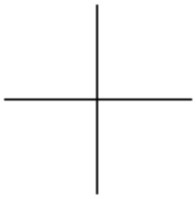
LG 11 Worksheet B (Ratios from Ratios)

If $\angle D$ is an angle in standard position, in which quadrants may $\angle D$ terminate if:

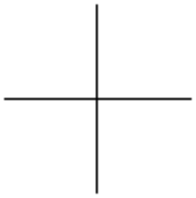
1. $\sec D < 0$



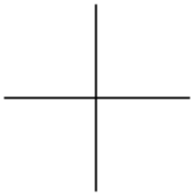
2. $\sin D > 0, 0 \leq D < 2\pi$



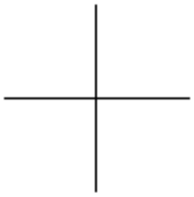
3. $\tan D < 0$ and $\pi \leq D < 2\pi$



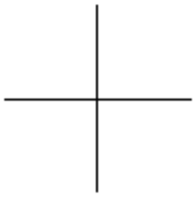
4. $\sec D > 0$ and $\frac{-\pi}{2} \leq D < \pi$



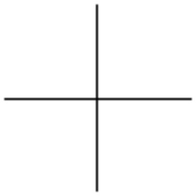
5. $\csc D < 0$ and $-\pi \leq D < \frac{\pi}{2}$



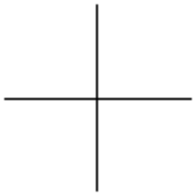
6. $\text{Sec } D > 0$ and $\text{Cot } D < 0$



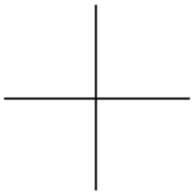
7. $\csc D > 0$ and $\cos D < 0$



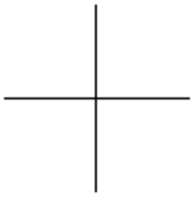
8. $\csc D > 0$ and $\frac{\pi}{2} \leq D < \frac{3\pi}{2}$



9. $\sin D < 0$ and $-\pi \leq D < \pi$

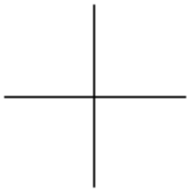


10. $\csc D > 0$ and $-\pi \leq D < 0$

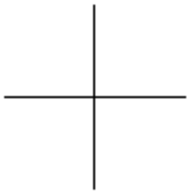


If $\sin A = \frac{-1}{3}$, find all the values of sec A when:

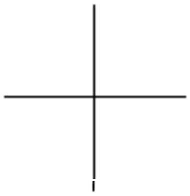
11. $0 \le A < 2\pi$



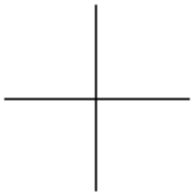
12. $-\pi \le A < \frac{\pi}{2}$



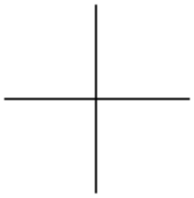
13. $\cot A > 0$



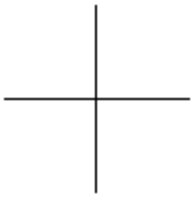
14. $\angle A$ terminates in quadrant IV



15. $\frac{-\pi}{2} \le A < \pi$



16. $\cos A < 0$ and $\tan A > 0$



Answer Key

1. II, III

2. I, II

3. IV

4. I, IV

5. III, IV

6. IV

7. II

8. II

9. III, IV

10. None

11. $\pm\frac{3}{2\sqrt{2}}$

12. $\pm\frac{3}{2\sqrt{2}}$

13. $\frac{-3}{2\sqrt{2}}$

14. $\frac{3}{2\sqrt{2}}$

15. $\frac{3}{2\sqrt{2}}$

16. $\frac{-3}{2\sqrt{2}}$

LG 11 Worksheet C (Trig. Review)

1. Find all the angles, A, that are co-terminal with $\frac{2\pi}{5}$ and in the domain: $-4\pi \le A < 4\pi$

2. Determine all the angles, A, that are co-terminal with $\frac{2\pi}{5}$.

3. Are the angles $\frac{7\pi}{12}$ and $\frac{43\pi}{12}$ co-terminal?

Convert the following to degrees exactly.

4. $\frac{-4\pi}{3}$ **5.** $\frac{11\pi}{4}$

Convert the following to degrees exactly.

6. 150° **7.** 200°

Convert to degrees to nearest tenth.

8. 4 **9.** $\frac{-5\pi}{6}$

Convert to radians to nearest hundredth.

10. 45° **11.** -340°

Find the exact value for each trigonometric ratio.

12. $csc\left(\frac{7\pi}{4}\right)$ **13.** $cot\left(\frac{-5\pi}{6}\right)$

14. $sec^2\left(\frac{5\pi}{4}\right)$ **15.** $tan^3(\pi)$

16. $\left(cos\left(\frac{-11\pi}{6}\right)\right)^2$ **17.** $sin^3\left(\frac{-3\pi}{2}\right)$

Find the value for each trigonometric ratio to 3 decimal places.

18. $cos\left(\frac{5\pi}{7}\right)$ **19.** $cot\left(\frac{3\pi}{8}\right)$

20. $csc\left(\frac{12}{7\pi}\right)$ **21.** $sec\left(\frac{-15\pi}{7}\right)$

22. If the terminal arm of $\angle B$ passes through the point (-2, -6). Find the exact values of the 6 trigonometric ratios of $\angle B$.

23. Find, to 2 decimal places, the length of the arc subtended by a central angle of $\frac{7\pi}{5}$ in a circle with a radius of 10 cm.

24. If an arc whose length is 20 m subtends a central angle of 130°, find the exact radius of the circle.

25. Find, to 1 decimal place, the arc length subtended by a central angle of 2.57 in a circle with a radius of 15 cm.

Given $cos A = \frac{2}{3}$, **find all values of csc A if:**

26. $0 \le A < 2\pi$

27. $\frac{-\pi}{2} \le A < 0$

28. Cot A > 0

29. $\angle A$ terminates in quadrant IV

30. $-2\pi \le A < \pi$

Answer Key

1.	$\frac{-18\pi}{5}, \frac{-8\pi}{5}, \frac{12\pi}{5}$	**18.**	-0.623
2.	$\frac{2\pi}{5} + 2n\pi, n \in I$	**19.**	0.414
3.	No	**20.**	1.927
4.	-240˚	**21.**	1.110
5.	495˚		
6.	$\frac{5\pi}{6}$		
7.	$\frac{10\pi}{9}$		
8.	229.2˚	**22.**	$\frac{-3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}, 3, \frac{-\sqrt{10}}{3}, -\sqrt{10}, \frac{1}{3}$
9.	-150˚	**23.**	43.98 cm
10.	0.79	**24.**	$\frac{360}{13\pi}$ m
11.	-5.93	**25.**	38.6 cm
12.	$-\sqrt{2}$	**26.**	$\frac{3}{\sqrt{5}}, \frac{-3}{\sqrt{5}}$
13.	$\sqrt{3}$	**27.**	$\frac{-3}{\sqrt{5}}$
14.	2	**28.**	$\frac{3}{\sqrt{5}}$
15.	0	**29.**	$\frac{-3}{\sqrt{5}}$
16.	$\frac{3}{4}$	**30.**	$\frac{3}{\sqrt{5}}, \frac{-3}{\sqrt{5}}$
17.	1		

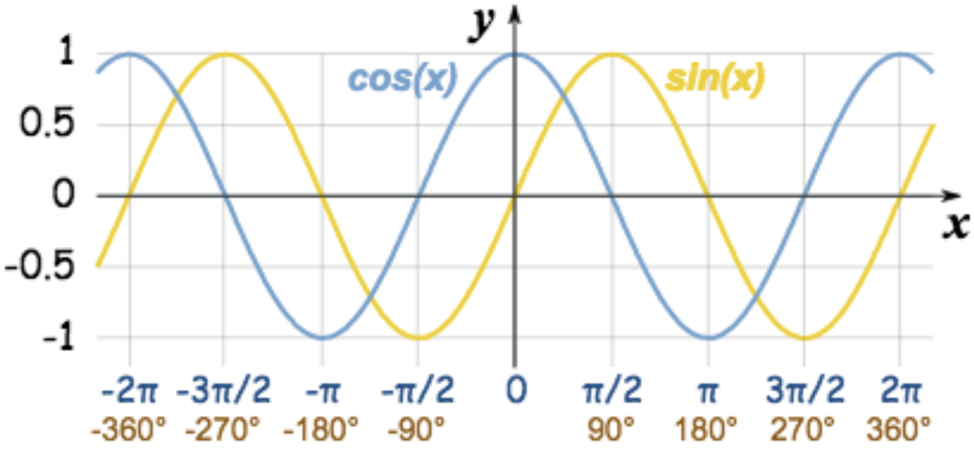
PRE-CALCULUS 12

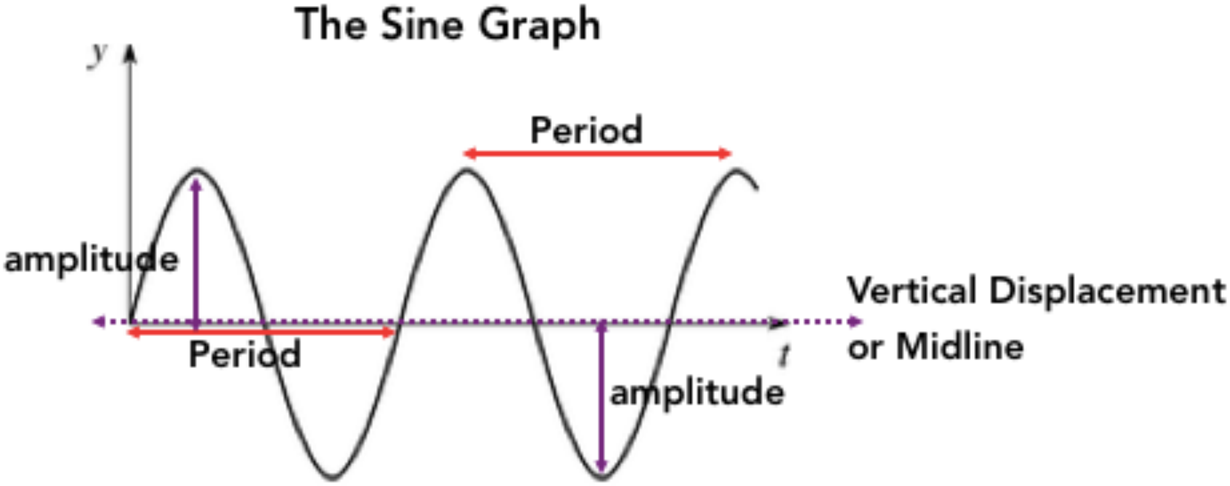
Seminar Notes

Learning Guides 12 & 13

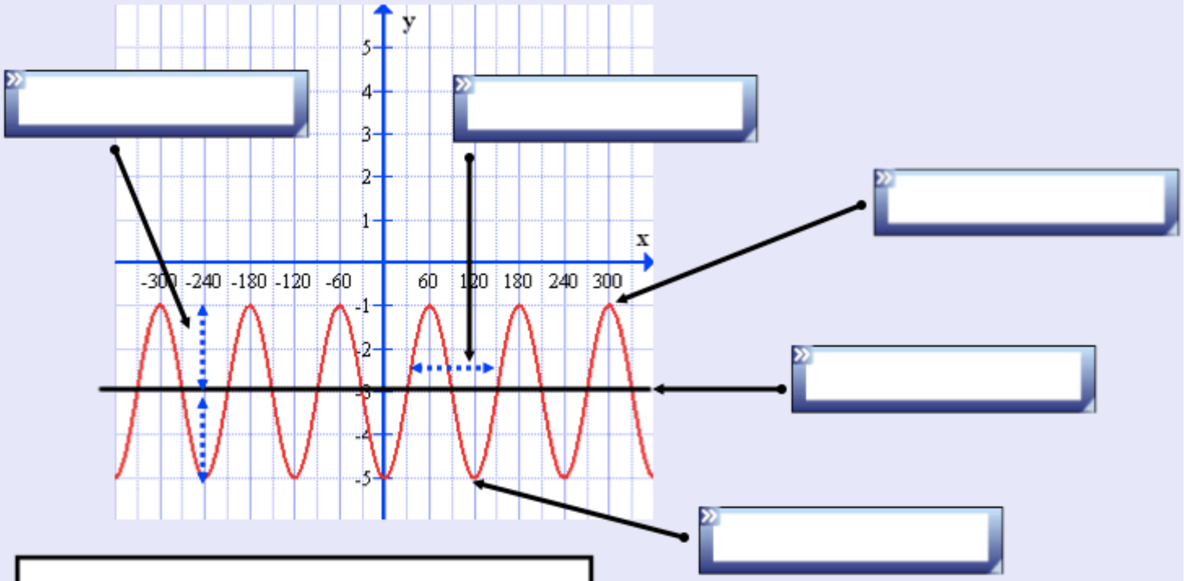
TRIGONOMETRIC FUNCTIONS & GRAPHS

The Sine & Cosine Graph





Sinusoidal Graphs and their characteristics



amplitude
period
maximum value
midline
minimum value

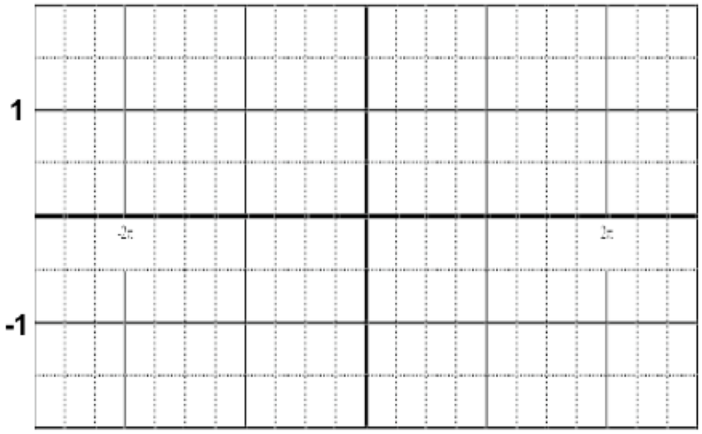
Write the following characteristics you see on the left in the correct boxes shown on the graph above.

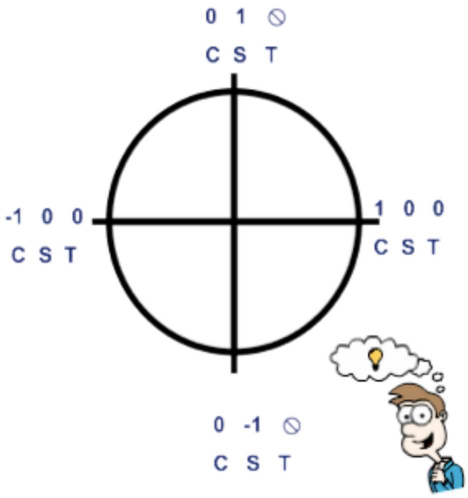
Topic 1

Graphing Sine & Cosine Functions

EXAMPLE 1 - Sketch the graph of y = Sin θ for $0 \le \theta \le 2\pi$

$\theta = 0$	$\theta = \frac{\pi}{2}$	$\theta = \pi$	$\theta = \frac{3\pi}{2}$	$\theta = 2\pi$



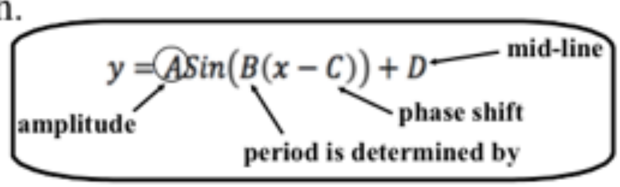


EXAMPLE 1b - Sketch the graph of Cos θ $0 \le \theta \le 2\pi$

$\theta = 0$	$\theta = \frac{\pi}{2}$	$\theta = \pi$	$\theta = \frac{3\pi}{2}$	$\theta = 2\pi$

EXAMPLE 2 - Amplitude of Sine and Cosine Functions

To understand **amplitude** you need to know the A,B,C,D's of a Sine or Cosine Graph.



Determine the amplitude of each function and then use the language of transformations to describe how each graph is related to $y = \sin x$ *or* $y = \cos x$.

a) $y = 4Cosx$

b) $y = \frac{1}{3}Sinx$

TRY Determine the amplitude of each function and then use the language of transformations to describe how each graph is related to $y = \sin x$ *or* $y = \cos x$.

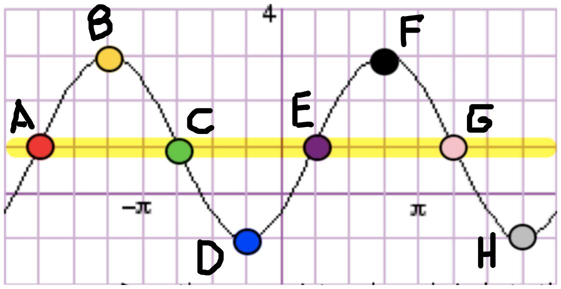
a) $y = 4 \sin x$

b) $y = - 0.5 \cos x$

Summary : Graphing Sin & Cos Functions

- **Positive Sin** ---> starts ***middle*** goes **up**
- **Negative Sin** ---> starts ***middle*** goes **down**
- **Positive Cos** ---> starts at the ***top*** goes **down**
- **Negative Cos** ---> starts at the ***bottom*** goes **up**

Place the correct letter in the appropriate column.



Sin +	Sin -	Cos +	Cos -

Summary:

- Vertical Displacement ***d*** ---> this is where you draw your "Mid-line".

ex. $Y = \text{Sin}\,x + 1$

- Amplitude **a** ---> this is the number you count up from the mid-line and then down from the mid-line.

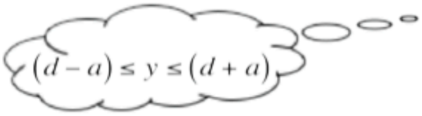
ex. $Y = 2\,\text{Sin}\,x + 1$

EXAMPLE 4 - Domain & Range of Sine and Cosine Functions

$$y = 3Sinx - 7$$

Domain:

Range:



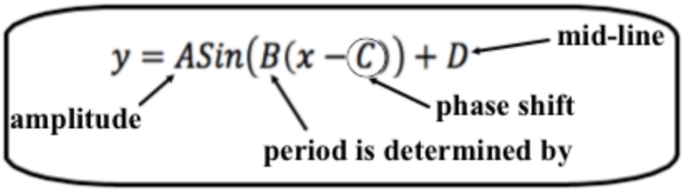
$(d-a) \le y \le (d+a)$

TRY: - Determine the domain & range of:

$$y = -4Cosx + 2$$

EXAMPLE 5 - Phase Shift

Comparing $y = \text{Sin}\,(x - c)$ to $y = \text{Sin}\, x$ &
$y = \text{Cos}\,(x - c)$ to $y = \text{Cos}\, x$

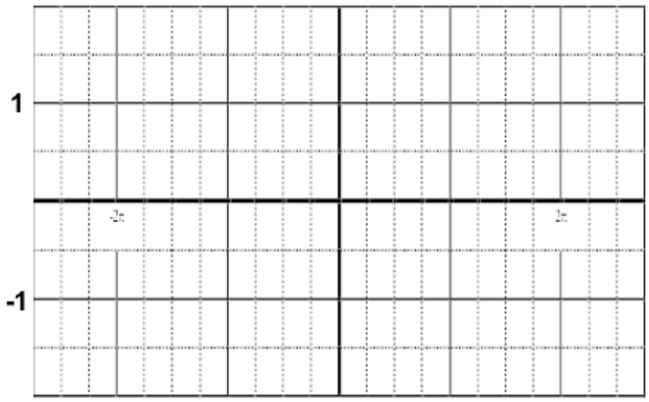


Graph the 3 functions below on same grid.

$y_1 = \text{Sin}\, x$

$y_2 = \text{Sin}\,(x - \frac{\pi}{2})$

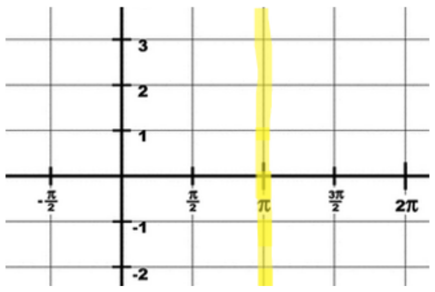
$y_3 = \text{Sin}\,(x + \frac{\pi}{3})$



What do you notice?

The phase shift formula is used to find the phase shift of a function. This determines where the sine/cosine wave starts. Phase Shift is a shift when the graph of the sine function and cosine function is shifted left or right from their usual position (x = 0), or we can say that in phase shift the function is shifted horizontally how far from the usual position

If you had the equation $y = 2\sin(x - \pi) + 1$



This is your Phase Shift you go to the right from 0 to π

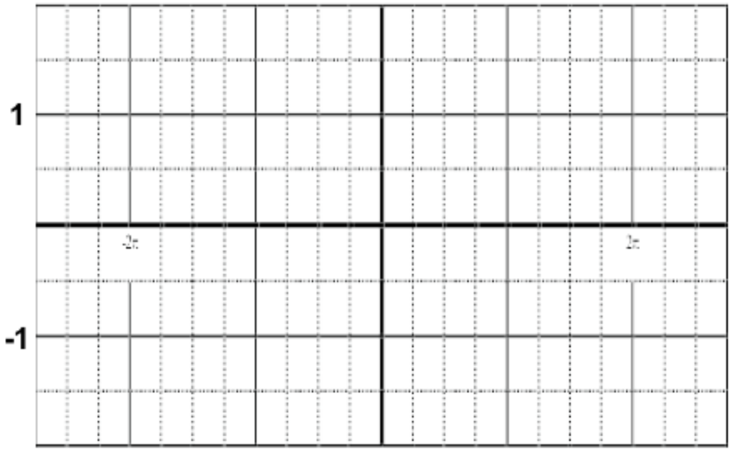
EXAMPLE 6 - Period Change

Comparing y = Sin (bx) to y = Sin x **and**
y = Cos (bx) to y = Cos x

$$y = \underset{\text{amplitude}}{A}Sin\left(\underset{\text{period is determined by}}{\textcircled{B}}(x - \underset{\text{phase shift}}{C})\right) + \overset{\text{mid-line}}{D}$$

Graph the 3 functions on same grid below.

$y_1 = \text{Cos } x$ $\quad$ $y_2 = \text{Cos } 4x$ $\quad$ $y_3 = \text{Cos } \frac{1}{2}x$



What do you notice?

Period Formula

The periods of y = Sin bx and y= Cos bx is determined by the formula:

$$P = \frac{\overset{360^\circ}{2\pi}}{b} \text{ or } b = \frac{\overset{360^\circ}{2\pi}}{P}$$

You need to find the P (period) when <u>graphing</u> and the b when <u>writing an equation</u>.

Calculate the period for the following functions:

	in Radians	in Degrees
a) $y = \text{Cos } 4x$		
b) $y = \text{Cos } \frac{1}{2}x$		

TRY: Determine the period of each function and then use the language of transformations to describe how each graph is related to ***y = sin x* or *y = cos x***.

a) $y = sin\ 5x$

b) $y = cos -\frac{1}{3}x$

Topic 2 Transformations of Sinusodial Functions

EXAMPLE 1 - Determine the amplitude, period, phase shift and vertical displacement of each function:

$a)\ \ y = 4\sin 2(x - \frac{\pi}{4}) + 5$

$y = a\text{Sin}\ b(x - c) + d$

a =
b =
c =
d =

$b)\ \ y = -6\cos 3(x - 45°) - 2$

Complete Assignment Questions #2 - #10

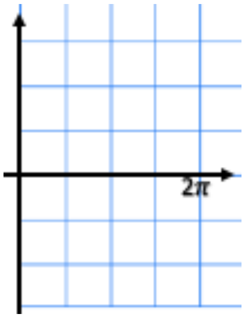
Assignment

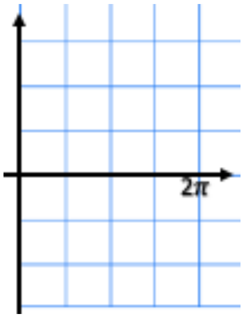
2. Determine the phase shift and the vertical displacement with respect to $y = \cos x$ for each function. Sketch a graph of each function.

a) $y = \cos (x - 30°) + 12$
b) $y = \cos \left(x - \frac{\pi}{3}\right)$
c) $y = \cos \left(x + \frac{5\pi}{6}\right) + 16$
d) $y = 4 \cos (x + 15°) + 3$
e) $y = 4 \cos (x - \pi) + 4$
f) $y = 3 \cos \left(2x - \frac{\pi}{6}\right) + 7$

4. State the amplitude of each periodic function. Sketch the graph of each function.

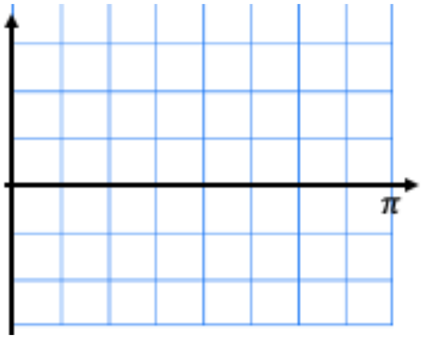
a) $y = 2 \sin \theta$ **b)** $y = \frac{1}{2} \cos \theta$

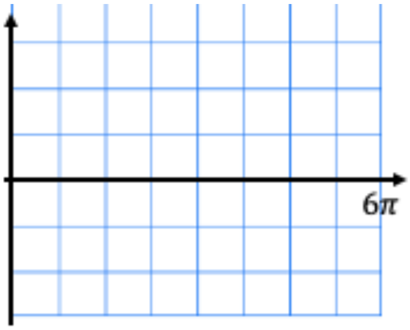




5. State the period for each periodic function, in degrees and in radians. Sketch the graph of each function.

a) $y = \sin 4\theta$ **b)** $y = \cos \frac{1}{3}\theta$





6. Match each function with its graph.

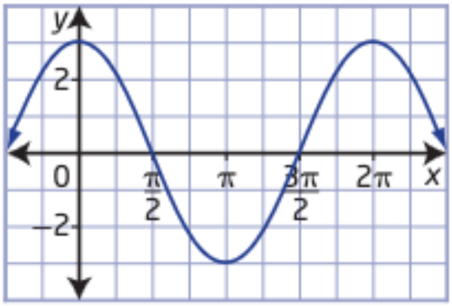
a) $y = 3 \cos x$

b) $y = \cos 3x$

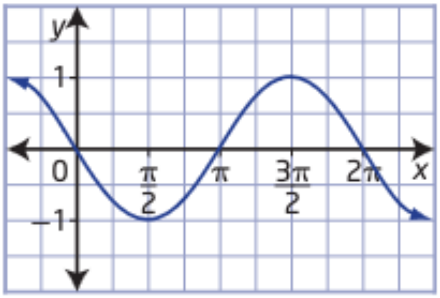
c) $y = -\sin x$

d) $y = -\cos x$

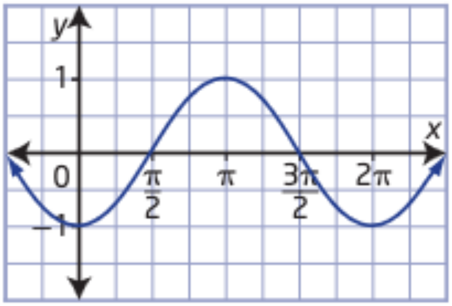
A



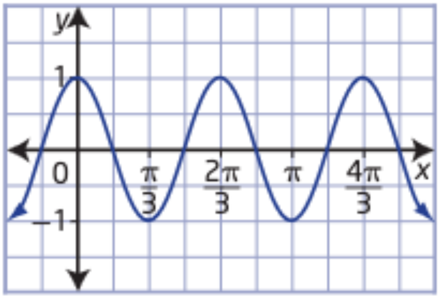
C



B



D



7. Determine the amplitude of each function. Then, use the language of transformations to describe how each graph is related to the graph of $y = \sin x$.

a) $y = 3 \sin x$ **b)** $y = -5 \sin x$

8. Determine the period (in degrees) of each function. Then, use the language of transformations to describe how each graph is related to the graph of $y = \cos x$.

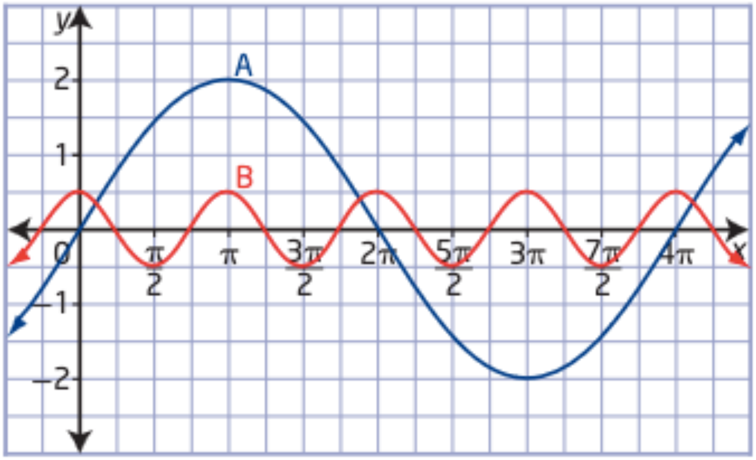
a) $y = \cos 2x$ **b)** $y = \cos (-3x)$

9. Without graphing, determine the amplitude and period of each function. State the period in degrees and in radians.

a) $y = 2 \sin x$ **b)** $y = -4 \cos 2x$

c) $y = \frac{5}{3} \sin \left(-\frac{2}{3}x\right)$ **d)** $y = 3 \cos \frac{1}{2}x$

10. a) Determine the period and the amplitude of each function in the graph.



A : ____________________

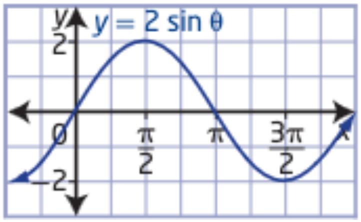
b) Write an equation in the form $y = a \sin bx$ or $y = a \cos bx$ for each function.

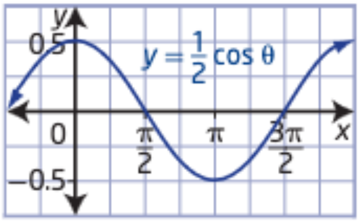
B: ____________________

Answer Key

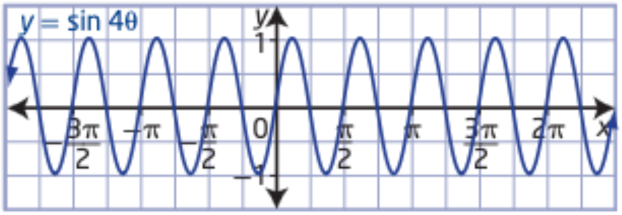
2. Phase shift: a) $R\ 30°$ b) $R\ \frac{\pi}{3}$ c) $L\ \frac{5\pi}{6}$ d) $L\ 15°$ e) $R\ \pi$ f) $R\ \frac{\pi}{12}$

4. a) 2 **b)** $\frac{1}{2}$

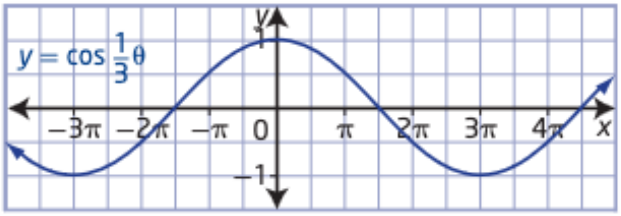




5. a) $\frac{\pi}{2}$ or 90°



b) 6π or 1080°



6. a) A **b)** D **c)** C **d)** B

7. a) Amplitude is 3; stretched vertically by a factor of 3 about the x-axis.

b) Amplitude is 5; stretched vertically by a factor of 5 about the x-axis and reflected in the x-axis.

8. a) Period is 180°; stretched horizontally by a factor of $\frac{1}{2}$ about the y-axis.

b) Period is 120°; stretched horizontally by a factor of $\frac{1}{3}$ about the y-axis and reflected in the y-axis.

9. a) Amplitude is 2; period is 360° or 2π.

b) Amplitude is 4; period is 180° or π.

c) Amplitude is $\frac{5}{3}$; period is 540° or 3π.

d) Amplitude is 3; period is 720° or 4π.

10. a) Graph A: Amplitude is 2 and period is 4π. Graph B: Amplitude is 0.5 and period is π.

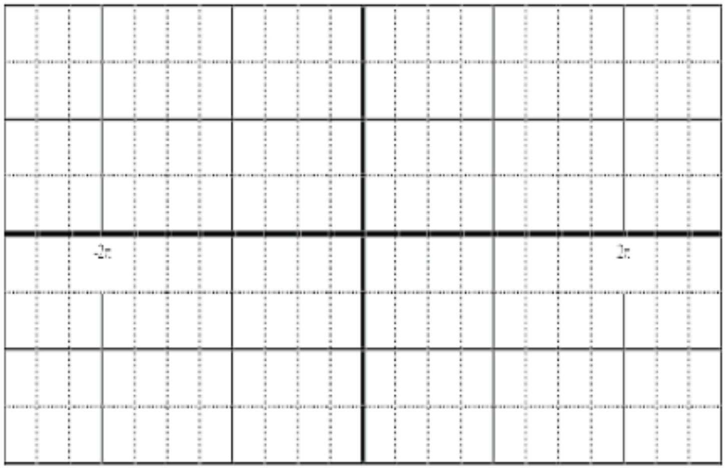
b) Graph A: $y = 2 \sin \frac{1}{2}x$; Graph B: $y = 0.5 \cos 2x$

Order for Graphing

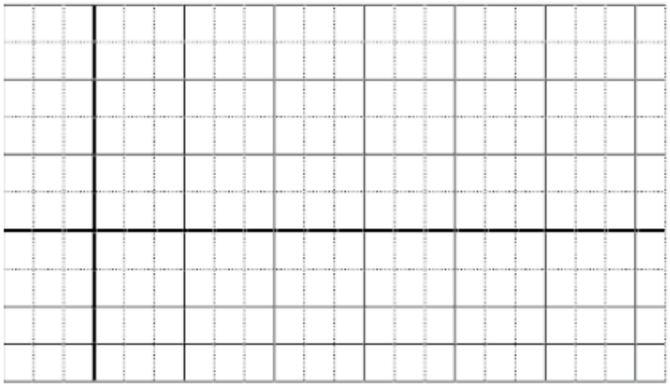
$$y = \boldsymbol{a}Sin\ \boldsymbol{b}(x - \boldsymbol{c}) + \boldsymbol{d}$$

1. Vertical displacement (***d***)
2. Amplitude (***a***)
3. Phase shift (***c***)
4. Period (***b***)

EXAMPLE 2 - **Graph the function y = 3Sin θ + 1 for two cycles. State the amplitude, the vertical displacement and the phase shift for the function.**



EXAMPLE 3 - **Graph the function** $f(x) = 2\text{Cos}\,(x - \frac{\pi}{3}) - 1$ **over two cycles. State the amplitude, the vertical displacement and the phase shift for the function.**



When you have a Phase Shift you must calculate a common denominator with your period and phase shift.

Vert. Displ. =

Amplitude =

Phase Shift =

a =

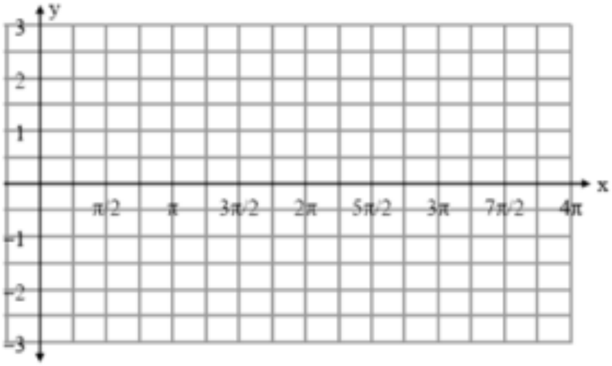
b =

c =

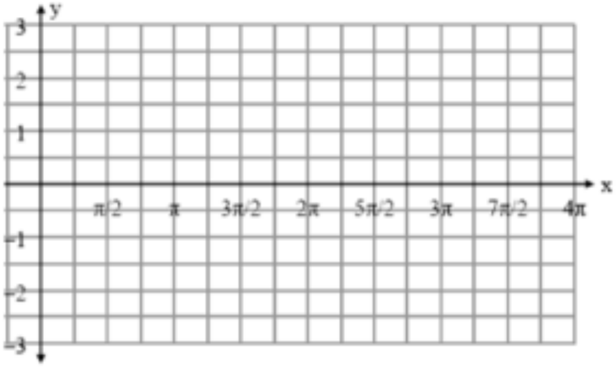
d =

TRY: Graph the following functions, then check your answer.

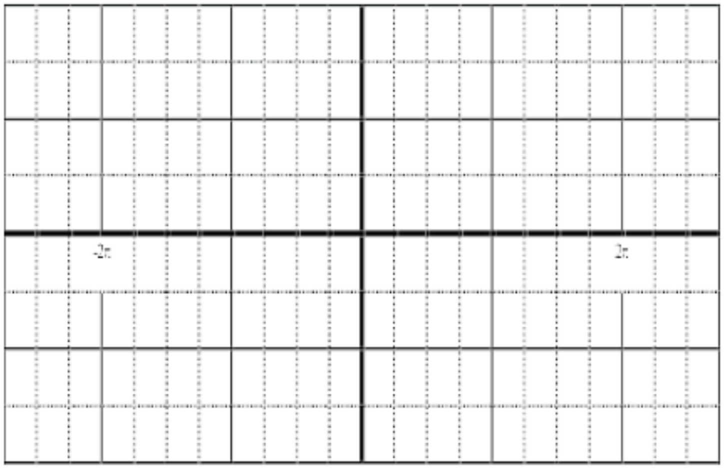
1) $y = 2\sin(2x) - 1$



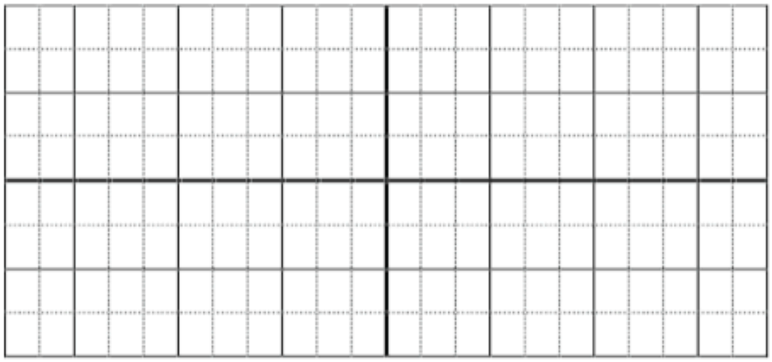
2) $y = -\sin(.5x) + 2$



TRY: **a) Graph** $y = 3\text{Cos}2(x - \frac{\pi}{2})$ over two cycles

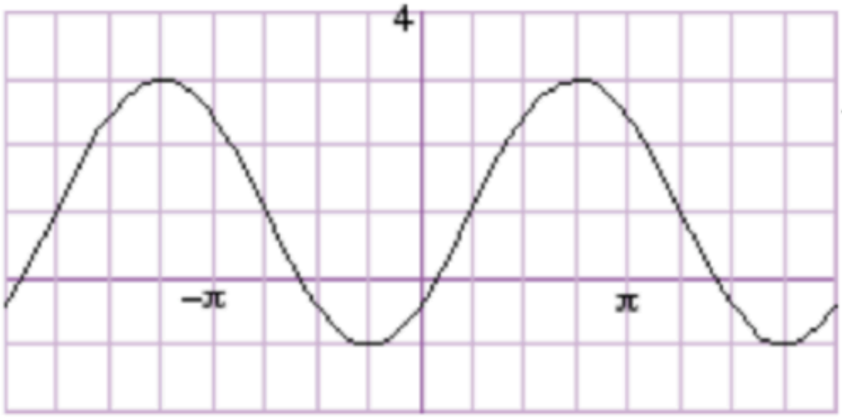


TRY: **a) Graph** $y = 3\text{Sin}(2x - \frac{\pi}{3})$ over two cycles



EXAMPLE 4 - For the sinusodial curve shown below, write:

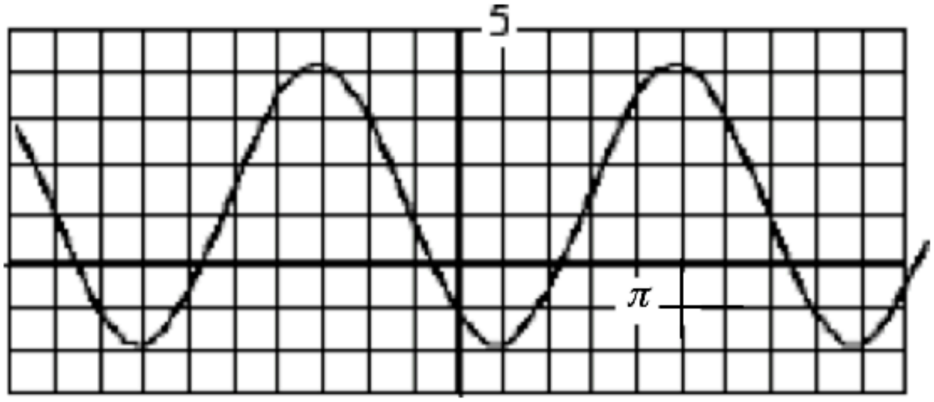
a) a possible cosine equation b) a possible sine equation



Sinusoidal means, you pick either Sine or Cosine

TRY: For the sinusodial curve shown below, write:

a) a possible cosine equation b) a possible sine equation

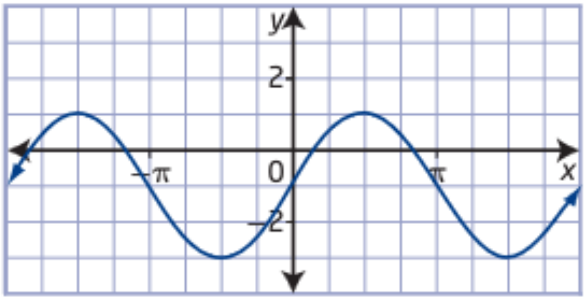


Complete Assignment Questions #15 - #17

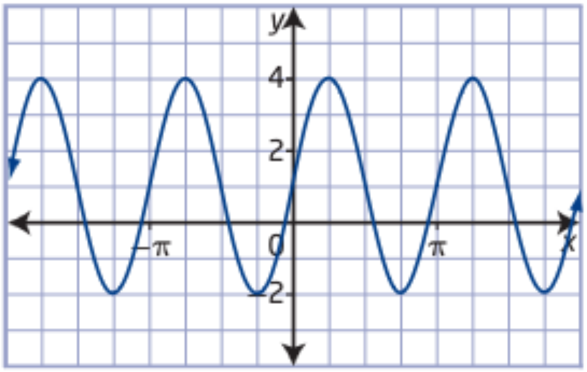
Assignment

15. Determine an equation in the form $y = a \sin b(x - c) + d$ for each graph.

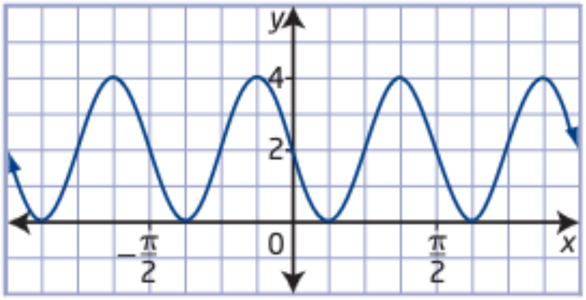
a)



b)

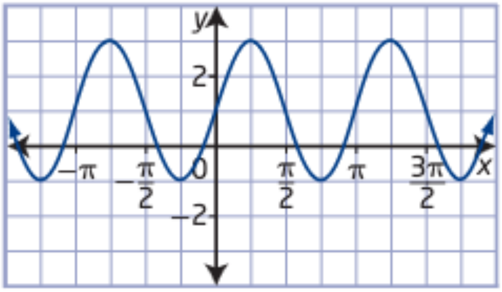


c)

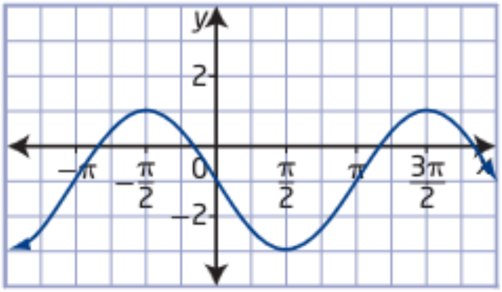


16. For each graph, write an equation in the form $y = a \cos b(x - c) + d$.

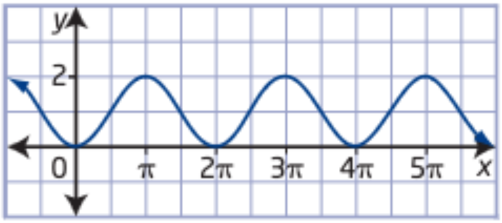
a)



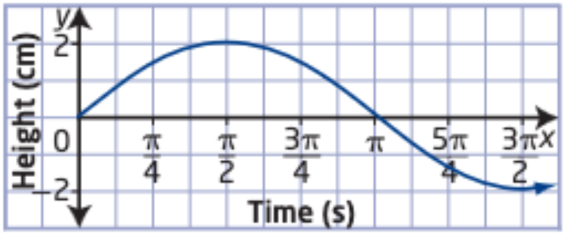
b)



c)



17. The piston engine is the most commonly used engine in the world. The height of the piston over time can be modelled by a sine curve. Given the equation for a sine curve, $y = a \sin b(x - c) + d$, which parameter(s) would be affected as the piston moves faster?



EXAMPLE 4b - The graph of y = cos x is transformed as described. Determine the value of the parameters **a, b, c,** and **d** for the transformed function. Write the equation for the transformed function in the form y = **a**cos **b**(x - **c**) + **d**.

a) vertical stretch by a factor of 4, horizontal stretch by a factor of 2, translated 5 units to the left and 7 units down.

TRY: **Write an equation for a cosine function with the following properties:**

Amplitude: 4
period: 10
phase shift: –2
vertical displacement: 1

TRY: **Write an equation for a sine function with the following properties:**

maximum: 16
minimum: 6
period: 7
phase shift: 3
vertical displacement: -5

Hint: *use the formula:*

$$A = \frac{M - m}{2}$$

where M is the maximum value
and m is the minimum value

EXAMPLE 5 - Graphing when b is in terms of π.

Graph the 3 functions on same grid below.

$y_1 = \text{Cos}2\pi x$

$y_2 = \text{Cos}\frac{2\pi}{3}x$

$a =$

$b =$ $p =$

$c =$

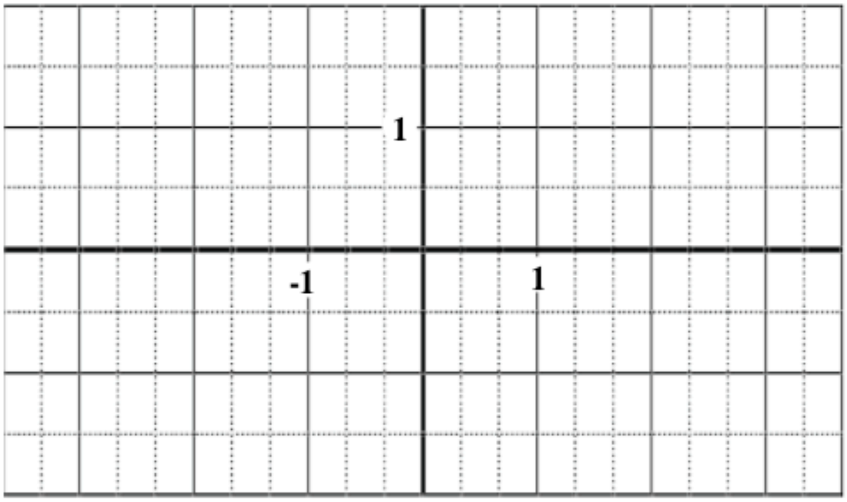
$d =$

$a =$

$b =$ $p =$

$c =$

$d =$



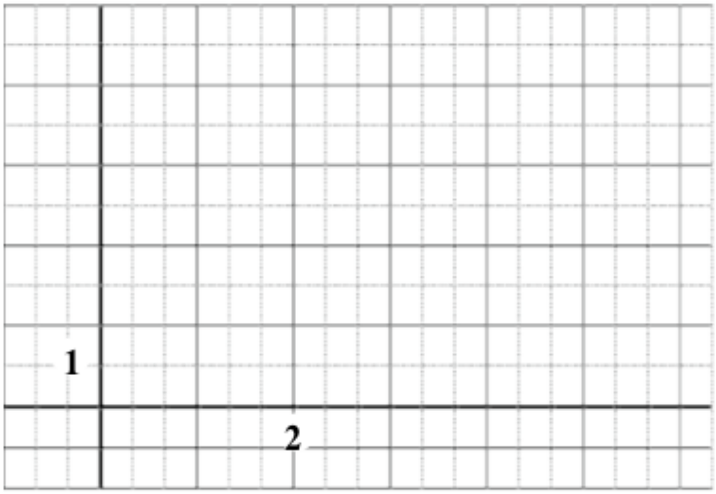
What do you notice that is different from this example to the previous examples?

Hint: x-axis

EXAMPLE 5b - Graphing when b is in terms of π.

$$y = 4Sin\frac{\pi}{2}(x+1)+2$$

Draw a sketch of this function on the axes below:



a =

b = **p =**

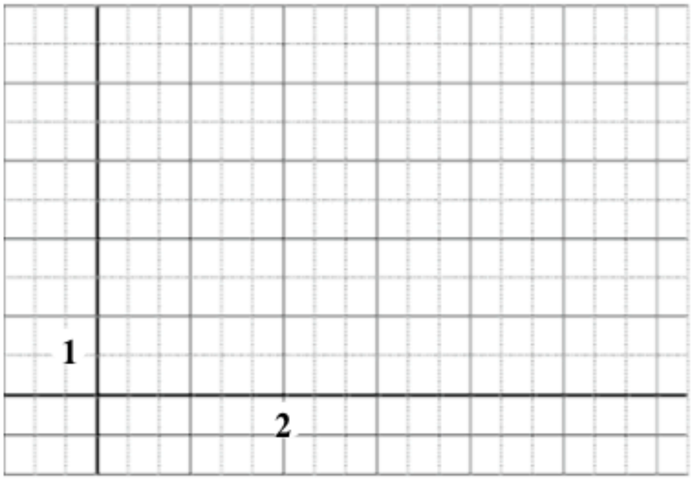
c =

d =

TRY - Graphing when b is in terms of π.

Graph the function : $y = 3\text{Cos}\,\frac{2\pi}{5}\,(x - 1) + 4$

Draw a sketch of this function on the axes below:



a =

b = **p =**

c =

d =

LG 12 Worksheet A (Trig Graphs)

1. If $y = -5\text{Sin}\ (2(x + \frac{\pi}{8})) - 7$, find the:

a) amplitude

b) vertical displacement

c) period

d. phase shift

e. max value of y

f) min value of y

2. If $y = 3\text{Cos}\ (6x - 12\pi) + 14$, find the:

a) amplitude

b) vertical displacement

c) period

d. phase shift

e. max value of y

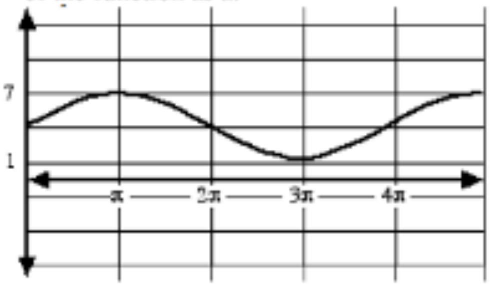
f) min value of y

3. If $y = -4\text{Cos}\ (2x - 8) - 10$, find the:

a) domain of the function.

b) range of the function.

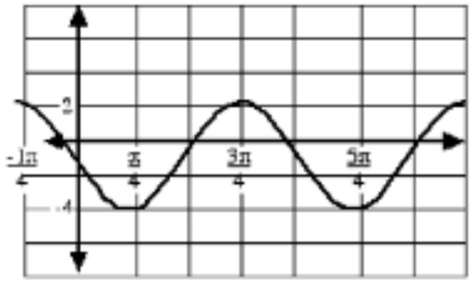
4. Given the sinusoidal graph below, write the equation of the function as a:



a) Sine function

b) Cosine function

5. Given the sinusoidal graph below, write the equation of the function as a:



a) Sine function

b) Cosine function

6. a) Find the maximum value of $f(x) = a\text{Sin}x + d$ where $a > 0$, $d > 0$.

b) Determine the period of $y = 8\text{Cos}\frac{2\pi}{15}x + 8$.

c) Determine the range of $y = 4\text{Cos}x - 2$.

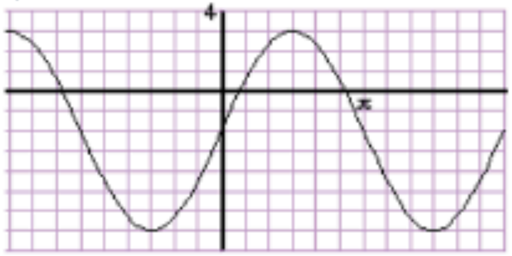
d) Determine the range of $y = -2\text{Sin}3x + 4$.

e) Determine the period of $f(x) = \frac{-1}{2}\text{Sin}\frac{x}{3}$.

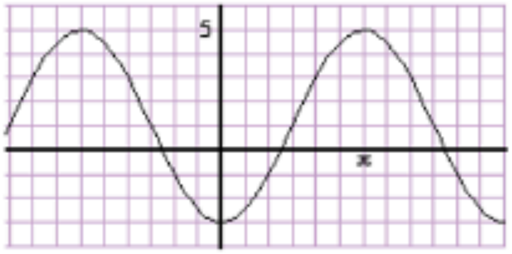
f) Find the range of $f(x) = b\text{Cos}ax - 2b$ where $a > 0$, $b > 0$.

7. Given the graphs below, determine an equation of the function.

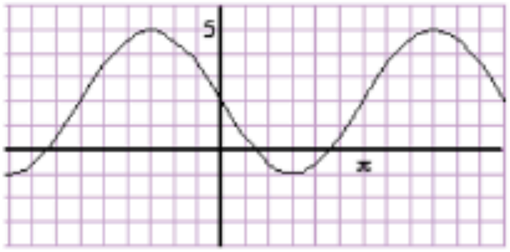
a)



b)

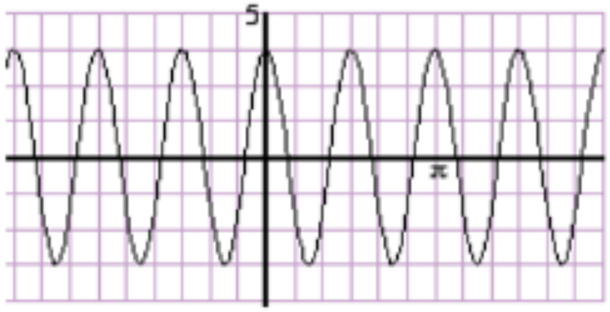


c)

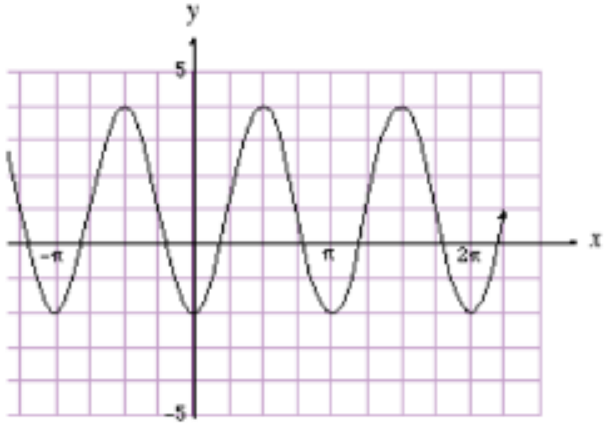


NOTE: one more question d) next page

d)



8. The graph below is a function that can be written in the form: y = aSin b(x – c) + d. Determine the values of a, b, c, d.



Answer Key

1. a. 5 d. $\frac{\pi}{8}$ left
b. -7 e. -2
c. π f. -12

2. a. 3 d. 2π right
b. 14 e. 17
c. $\frac{\pi}{3}$ f. 11

3. a) All real #'s
b) $-14 \le y \le -6$

4. a. $y = 3\text{Sin}\,\frac{1}{2}(x + 0) + 4$
b. $y = 3\text{Cos}\,\frac{1}{2}(x - \pi) + 4$

5. a. $y = 3\text{Sin}\,2(x - \frac{\pi}{2}) - 1$
b. $y = 3\text{Cos}\,2(x + \frac{\pi}{4}) - 1$

6. a. a + d d. $2 \le y \le 6$
b. 15 e. 6π
c. $-6 \le y \le 2$ f. $-3b \le y \le -b$

note: *there are many different possibilities for the answers to questions 7 and 8.*

7. a) y = 5Sin x – 2
b) y = -4Cos x + 1
c) y = -3Sin x + 2
d) y = 3Cos 4x

8. $a = 3, b = 2, c = \frac{\pi}{4}, d = 1$

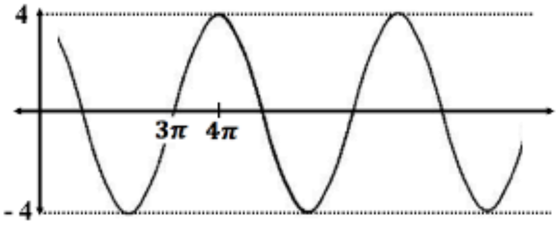
or

$a = -3, b = 2, c = \frac{-\pi}{4}, d = 1$

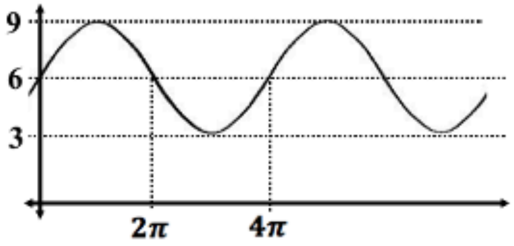
LG 12 Worksheet B (Review Trigonometric Graphing)

1. Determine the amplitude and the period, in both degrees and radians, for $y = -5\cos\left(\frac{1}{2}x\right)$.

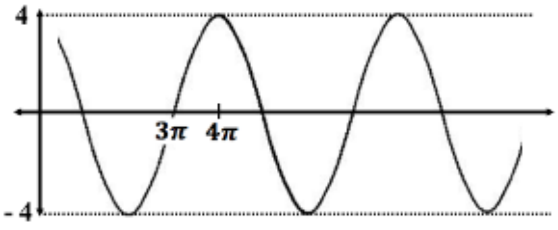
2. Determine the amplitude and the period for the function graphed below:



3. Given the sinusoidal curve graphed below, write its equation in the form $y = a\sin b(x-c)+d$ & $y = a\cos b(x-c)+d$.



4. Given the sinusoidal curve graphed below, write its equation in the form $y = a\sin b(x-c)+d$ & $y = a\cos b(x-c)+d$.



5. Given the function $y = -5\cos\left(2x + \frac{\pi}{6}\right) - 7$ find:

a. amplitude

b. period

c. vertical displacement

d. phase shift

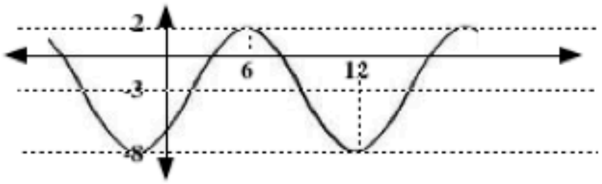
e. domain

f. range

g. minimum value of y

h. maximum value of y

6. Find the amplitude, period, and vertical displacement of the sinusoidal curve graphed below.



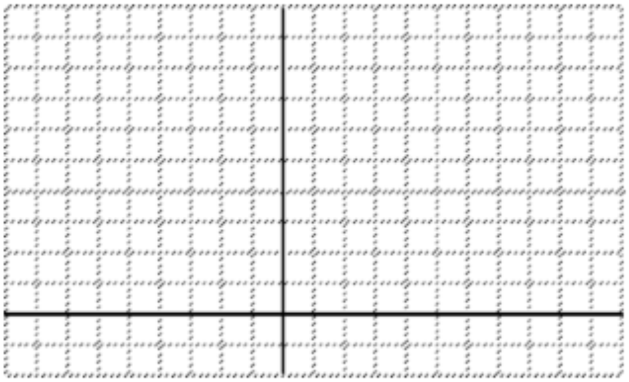
a. amplitude

b. period

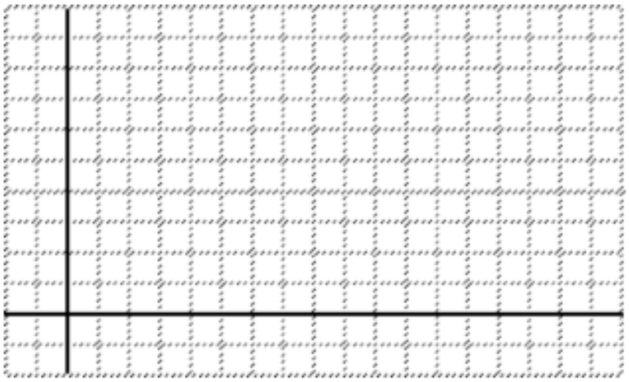
c. vertical displacement

7. Write an equation of the cosine function with amplitude 0.5, period 10π, phase shift $\frac{\pi}{6}$ to the right, and vertical displacement -9.

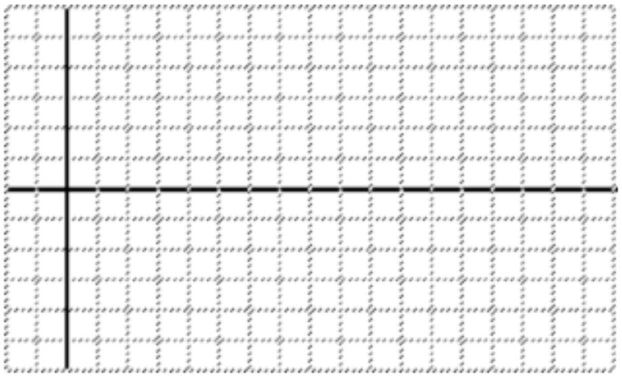
8. Determine the equation of the sinusoidal function with a minimum at (-8, 3) and the nearest maximum to the right at (0, 9). Write the equations in the form $y = a\sin b(x-c)+d$ or $y = a\cos b(x-c)+d$.



9. Sketch the graph of the following function. $y = -3\cos(2x-90^\circ)+5$, $0^\circ \le x < 360^\circ$



10. Sketch the graph of the following function. $y = 4\sin\left(3x-\frac{\pi}{2}\right)+1$ for 2 cycles.



Answer Key

1. amp. = 5, period = 4π or 720°

2. amp. = 4, period = 4π

3. $y = -3\text{Sin}\,\frac{1}{2}(x-3\pi)+6$ & $y = -3\text{Cos}\,\frac{1}{2}(x-4\pi)+6$

4. $y = 4\text{Sin}\,\frac{1}{2}(x-2\pi)+0$ & $y = 4\text{Cos}\,\frac{1}{2}(x-4\pi)+0$

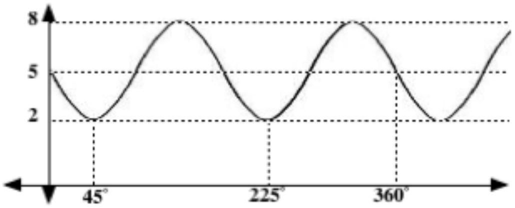
5. amplitude = 5
period = π
vert. displ. = -7
ph. shift = $\frac{\pi}{12}$ left
domain: all real #'s
range: $-12 \le y \le -2$
max y: -2
min. y: -12

6. amplitude = 5
period = 12
vert. displ. = -3

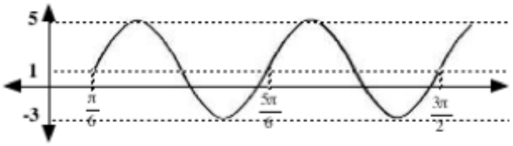
7. $y = 0.5\cos\frac{1}{5}\left(x-\frac{\pi}{6}\right)-9$

8. $y = 3\sin\frac{\pi}{8}(x+4)+6$ & $y = 3\cos\frac{\pi}{8}(x-0)+6$

9.



10.



LEARNING GUIDE 13

Tangent Functions

$y = tanx$

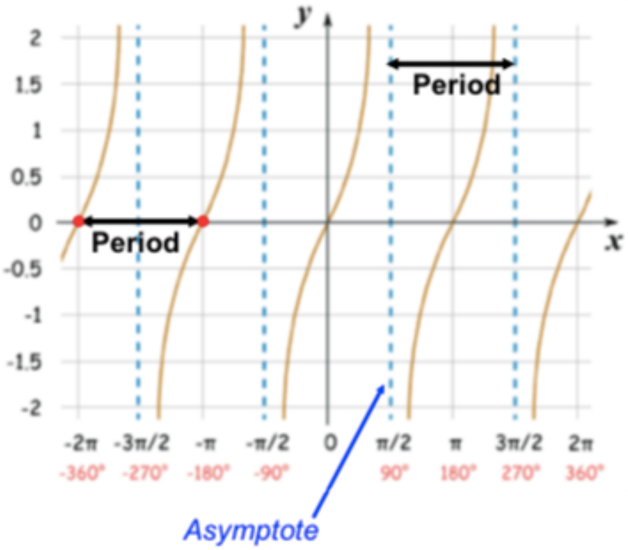
Note:

⇒ **there is no amplitude (*a*) or vertical displacement (*d*)**

⇒ **there is only period which is referred as *k*, and phase shift (*c*)**

⇒ **the period is** $p = \frac{\pi}{k}$ **NOT** $\frac{2\pi}{b}$ **as in Sin/Cos**

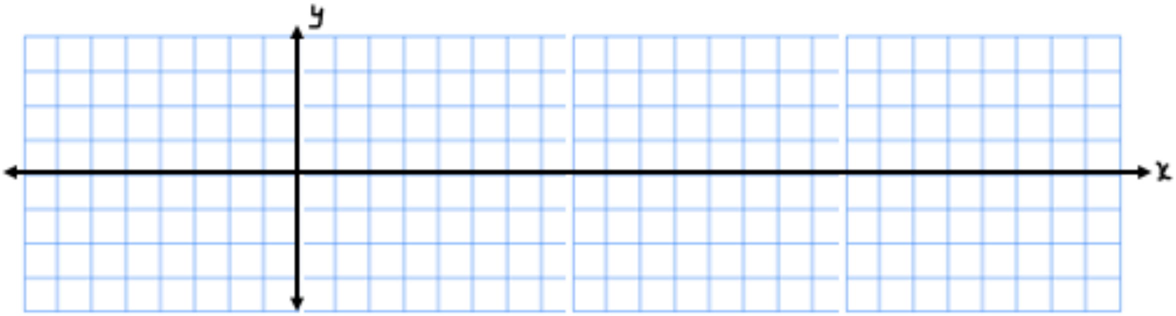
☆ ⇒ **asymptotes are important**



CHARACTERISTICS OF THE TANGENT FUNCTION & IT'S CONNECTION TO SINE and COSINE

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$sin\theta$																	
$cos\theta$																	
$tan\theta$																	

Use the above information to graph $y = tan\theta$. $(Note\ \theta\ and\ x\ are\ interchangeable)$



Topic 1

The Tangent Function

EXAMPLE 1 - Graph y = tan 2x

a) state the domain and range

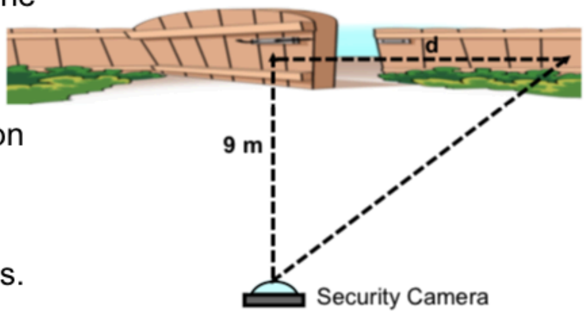
b) find the value of:

i) $\frac{\pi}{8} =$

ii) $-\frac{\pi}{4} =$

EXAMPLE 2 - Model a Problem Using the Tangent Function

A security camera scans a long a fence. The camera is 9 m from the fence and makes one complete rotation every 40 s.



a) Determine the tangent function that represents distance, d and time, t.

b) What is the distance at t = 6 s.

c) what happens when t = 10 s.

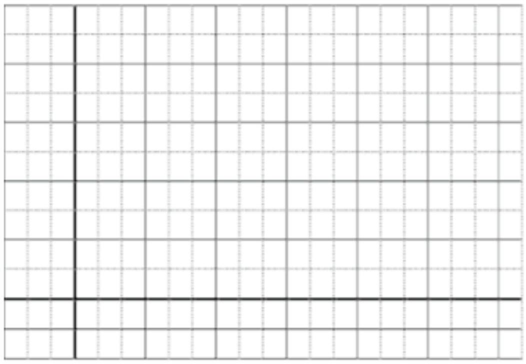
Topic 2 Equations & Graphs of Trig. Functions

EXAMPLE 1 - Modelling Real Situations

Tides are the *periodic* rise and fall of the water in the oceans. We can use a sinusoidal curve as a model for this periodic motion. The following equation models the height, *h*, in metres, of the water at time *t* , (after midnight):

$$h = 3\text{Cos}\frac{2\pi}{12.4}(t - 4.5) + 5$$

Draw a sketch of this function on the axes below:



a) Estimate, to the nearest tenth of a metre, the depth of the water at 2:45 pm

Hint: *Change the minutes to decimals by dividing the minutes by 60*

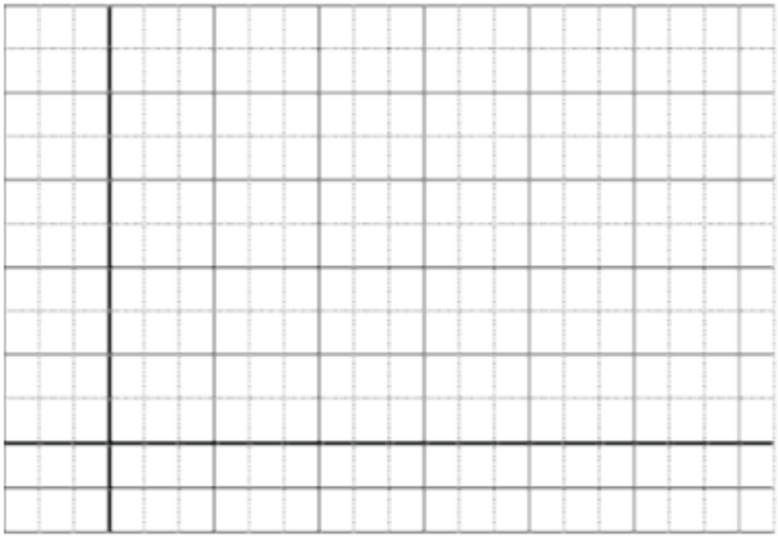
b) Estimate, to the nearest minute, one of the times when the water is 2.5 m deep on the day represented by the equation.

Hint: *Change the decimal to minutes by multiplying the decimal part by 60*

TRY: - Model a Problem Using a Sinusoidal Function

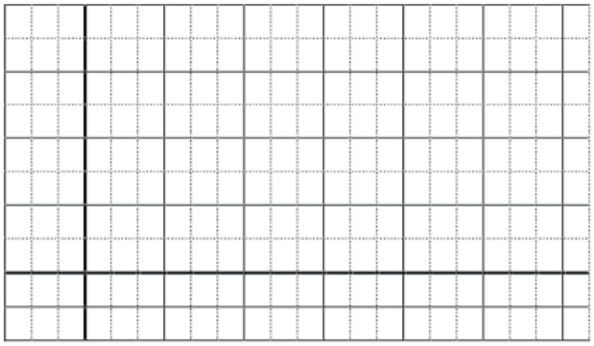
The high and low tides for a small fishing dock are as listed: low 10 at 3am, high 18 at 12:30pm, and low 10 at 10pm. A ship making a delivery needs a water depth of 15 feet to get his boat to the dock.When is the earliest he will be able to deliver his goods?

Draw a sketch of this function on the axes below:



EXAMPLE 2 - Model a Problem Using a Sinusoidal Function

A Ferris wheel has a radius of 20 m. It rotates once every 40 seconds. Passengers get on at its lowest point, which is 2 m above ground level. Suppose you get on at its lowest point and the wheel starts to rotate.



a) Graph how your height above the ground varies during the first cycle.

b) Write the equation that expresses your height as a function of the elapsed time.

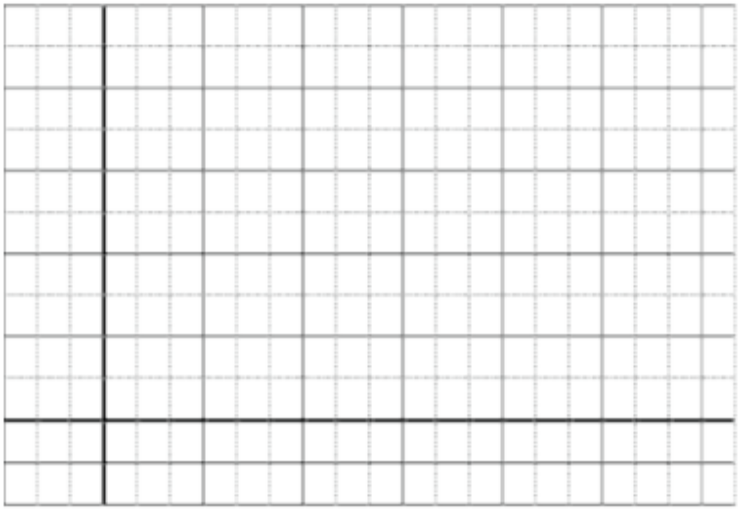
c) Estimate your height above the ground after 45 s.

d) Estimate the first time when your height is 35 m above the ground.

TRY: - Model a Problem Using a Sinusoidal Function

A Ferris Wheel that is 60 feet in diameter makes a revolution every 80 seconds. If the center of the wheel is 35 feet above the ground, how long does it take for a rider (who starts his journey at the bottom of the wheel) to reach a height of 50 feet?

Draw a sketch of this function on the axes below:



LG 13 Worksheet A (Applied Trig. Graphs)

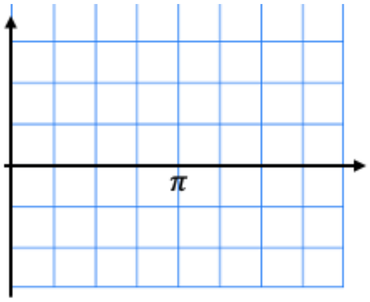
1. At a seaport, the depth of the water *h* in metres at time *t* = 1 hour during a certain day is given by this formula:

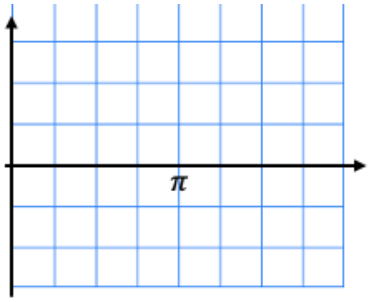
$$h = 2.4\text{Cos}\,\frac{2\pi}{12.4}(t - 5) + 4.2$$

Assume that when *t* = 0, it is midnight. Use *DESMOS* to determine how long after midnight (the first two times) that the water is 5 m deep.

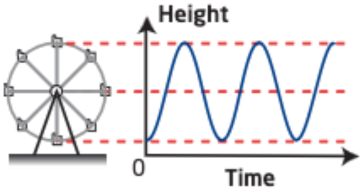
2. Graph the following tangent functions.

$a)\ tan\frac{1}{2}x$ $b)\ tan\frac{3}{4}x$





3. A Ferris wheel with a radius of 10 m rotates once every 60 s. Passengers get on board at a point 2 m above the ground at the bottom of the Ferris wheel. A sketch for the first 150 s is shown.



a) Write an equation to model the path of a passenger on the Ferris wheel, where the height is a function of time.

b) If Emily is at the bottom of the Ferris wheel when it begins to move, determine her height above the ground, to the nearest tenth of a metre, when the wheel has been in motion for 2.3 min.

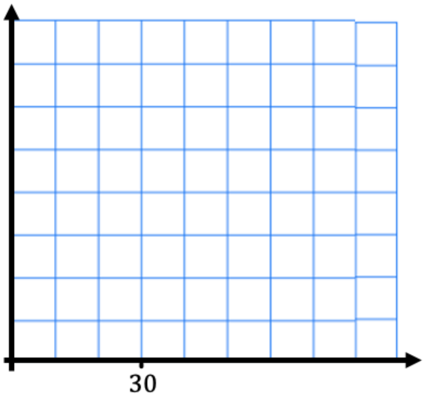
4. The height, h, in metres, above the ground of a rider on a Ferris wheel after t seconds can be modelled by the sine function $h(t) = 12 \sin \frac{\pi}{45}(t - 30) + 15$.

a) Graph the function using graphing technology.

b) Determine the maximum and minimum heights of the rider above the ground.

c) Determine the time required for the Ferris wheel to complete one revolution.

d) Determine the height of the rider above the ground after 45 s.



5. The number of hours of daylight, L, in Lethbridge, Alberta, may be modelled by a sinusoidal function of time, t. The longest day of the year is June 21, with 15.7 h of daylight, and the shortest day is December 21, with 8.3 h of daylight.

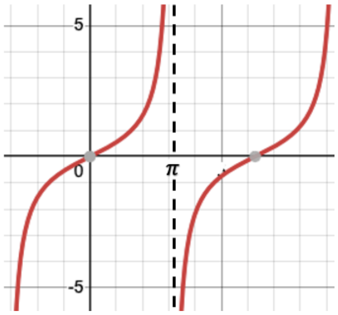
a) Determine a sinusoidal function to model this situation.

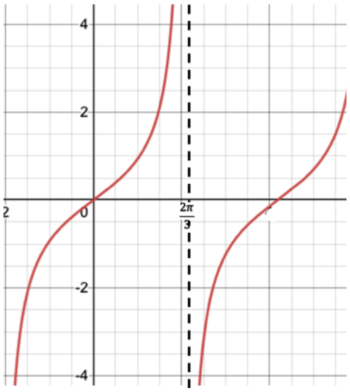
b) How many hours of daylight are there on April 3?

Answer Key

1. 2.57 hrs, 7.43 hrs

2.



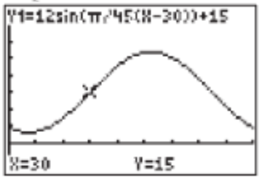


3.

a) $h = -10 \cos \frac{\pi}{30}t + 12$, where t represents the time, in seconds, and h represents the height of a passenger, in metres, above the ground

b) 15.1 m

4. a)



Y1=12sin(π/45(X-30))+15

X=30 Y=15

b) maximum height: 27 m, minimum height: 3 m

c) 90 s

d) approximately 25.4 m

5. a) $L = -3.7 \cos \frac{2\pi}{365}(t + 10) + 12$

b) approximately 12.8 h of daylight

PRE-CALCULUS 12

Seminar Notes

Learning Guides 14 - 15

TRIGONOMETRIC IDENTITIES

Review

Simplify the following:

1. $a\frac{b}{c}\frac{c}{a}\frac{1}{b}$

2. $a\frac{b^2}{a^2}\frac{1}{b}$

3. $\frac{1}{\frac{1}{a}}$

4. $\frac{\frac{a}{b}}{\frac{c}{d}}$

5. $\frac{a+\frac{a}{b}}{b+\frac{a}{b}}$

6. $\frac{a}{b}+\frac{c}{b}$

7. $\frac{a}{b}+\frac{b}{c}$

8 Rules for Success with Trig. Identities

1. Change $Sin \text{ to } s$ $\quad Csc \text{ to } \frac{1}{s}$
   $Cos \text{ to } c$ $\quad Sec \text{ to } \frac{1}{c}$
   $Tan \text{ to } \frac{s}{c}$ $\quad Cot \text{ to } \frac{c}{s}$

   Example:
   $Secx + SinxCosx \Rightarrow \frac{1}{c} + sc$

2. Tidy-up - watch for cancelling

   Example:
   $s \bullet \frac{c}{s}$

3. Clear complex fractions

   Example:
   $\frac{Sinx}{Tanx} \Rightarrow \frac{s}{s/c}$

4. Fraction Operation

   Example 1:
   $\frac{1}{1+s} + \frac{1}{1-s}$

   Example 2:
   $\frac{c}{s} - s$

5. Factoring

   Example 1:
   $c^2 + 2c + 1$

   Example 2:
   $s^2 - 1$

6. Foil

   Example:
   $(1+s)(1-s)$

7. Pythagorean Rules

   Example:
   $s^2 + c^2 = 1$

8. OMG "I'm Stuck"

   Example:
   Conjugate

Topic 1 Proving Trigonometric Identities

EXAMPLE 1 - Using Algebra

1.	$\operatorname{Sin}\theta \operatorname{Sec}\theta \operatorname{Cot}\theta$	=	1

2.	$\operatorname{Sec}\theta$	=	$\operatorname{Tan}\theta \ \operatorname{Csc}\theta$

3. $\dfrac{Cot\ \theta}{Csc\ \theta}$	=	$Cos\ \theta$

4. $Sec\ \theta\ (1 + Cos\ \theta)$	=	$1 + Sec\ \theta$

5. $\dfrac{Sin\ \theta + Cos\ \theta}{Sin\ \theta}$	=	$1 + Cot\ \theta$

6. $\dfrac{1 + Tan\ \theta}{1 + Cot\ \theta}$	=	$\dfrac{1 - Tan\ \theta}{Cot\ \theta - 1}$

There are three more important basic identities that are based on the Pythagorean formula.

Pythagorean Identities:

$\text{Sin}^2\theta + \text{Cos}^2\theta =$

$\text{Sin}^2\theta =$

$\text{Cos}^2\theta =$

7. $1 - Cos^2\ \theta$ =	$Cos^2\ \theta\ \ Tan^2\ \theta$

8. $(Sec\ \ \theta - 1)(Sec\ \ \theta + 1)$ =	$\dfrac{1}{Cot^2\ \theta}$

EXAMPLE 4 - **NON-PERMISSIBLE VALUES:**

Determine the non-permissilbe values of x, in radians.

$Csc\, x$	$\dfrac{Sec\, x}{Sin\, x+1}$
Solution: ☛ $\csc x = \dfrac{1}{\sin x} \therefore \sin x \neq 0$ Unit Circle ☛ $\sin x = 0\ is\ at\ 0\ \&\ \pi$ ☛ $non - permissible\ are: x \neq \pi n,\ n \in I$	

TRY: Determine the non-permissilbe values of x, in radians.

$$\frac{\tan x}{1-\sin x}$$

EXAMPLE 5 - Proving Trig Identities using the Conjugate

1. $\frac{Sin\ \theta}{1 + Cos\ \theta}$ =	$\frac{1 - Cos\ \theta}{Sin\ \theta}$

TRY: Prove the following Trig, Identities

1. $\cos x(\csc x + \tan x) = \cot x + \sin x$

2. ☆ $\csc x + \cot x = \frac{\sin x}{1 - \cos x}$

3. $\cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

4. $\frac{\csc^2 x - 1}{\csc^2 x} = \cos^2 x$

TRY: Prove the following Trig, Identities

5. $\sec x - \tan x \sin x = \cot x \sin x$

6. $\dfrac{\cos^4 x - \sin^4 x}{\sin^2 x} = \cot x - 1$

LG 14 Worksheet A (Trig Proofs)

1. Show that the LHS is equal to the RHS

a. $\frac{1}{s} + \frac{1}{c} = \frac{c+s}{cs}$

b. $\frac{1}{c} - \frac{1}{s} = \frac{s-c}{cs}$

c. $\frac{1}{s} \text{ x } \frac{1}{c} = \frac{1}{cs}$

d. $\frac{1}{s} \div \frac{1}{c} = \frac{c}{s}$

e. $\frac{\frac{1}{s}}{\frac{1}{c}} = \frac{c}{s}$

f. $\frac{\frac{1}{s}}{\frac{1}{s^2}} = s$

g. $\frac{\frac{1}{c^2}}{\frac{1}{c}} = \frac{1}{c}$

h. $\frac{\frac{1}{c}}{\frac{1}{s}} = \frac{s}{c}$

h. $\frac{\frac{1}{c}}{\frac{1}{s}} = \frac{s}{c}$

i. $\frac{c^2s}{cs^2} = \frac{c}{s}$

j. $\frac{2c^2s^2}{2cs} = cs$

k. $(c + s)^2 = c^2 + s^2 + 2sc$

l. $(c + s)(c - s) = c^2 - s^2$

2. Show the LHS = RHS (Hint: factor!)

a. $\frac{c^2-1}{c-1} = c+1$

b. $\frac{c^2+2c+1}{c^2-1} = \frac{c+1}{c-1}$

c. $\frac{cs+c^2}{s+c} = c$

d. $\frac{cs+s^2}{c^2+sc} = \frac{s}{c}$

e. $\frac{c+s}{c^2-s^2} = \frac{1}{c-s}$

f. $\frac{c^2-s^2}{c-s} = c+s$

3. Show the LHS = RHS (Hint: complex fractions!)

a. $\frac{\frac{1}{s} + \frac{1}{c}}{\frac{1}{s} - \frac{1}{c}} = \frac{c+s}{c-s}$

b. $\frac{\frac{1}{s} + 1}{\frac{1}{s} - 1} = \frac{1+s}{1-s}$

b. $\frac{\frac{1}{s} + 1}{\frac{1}{s} - 1} = \frac{1+s}{1-s}$

c. $\frac{1 - \frac{1}{c}}{1 + \frac{1}{c}} = \frac{c-1}{c+1}$

d. $\frac{s - \frac{1}{c}}{s + \frac{1}{c}} = \frac{sc-1}{sc+1}$

4. Show the LHS = RHS (Hint: add fractions!)

a. $\frac{1}{c}+\frac{1}{c+1} = \frac{2c+1}{c(c+1)}$

b. $\frac{1}{c}-\frac{1}{c+s} = \frac{s}{c(c+s)}$

c. $\frac{1}{c+1}+\frac{1}{c-1} = \frac{2c}{(c+1)(c-1)}$

5. Show the LHS = RHS (Hint: complex & factor!)

a. $\dfrac{1+\frac{s}{c}}{1+\frac{c}{s}} = \frac{s}{c}$

b. $\dfrac{c}{\frac{1}{c}-1} = \frac{c^2}{1-c}$

c. $\dfrac{\frac{1}{c}}{s}-\frac{s}{c} = \frac{1-s^2}{cs}$

d. $\dfrac{\frac{s}{c}}{1+\frac{s}{c}} = \frac{s}{c+s}$

e. $\dfrac{\frac{1}{s}-\frac{1}{c}}{\frac{c}{s}} = \frac{c-s}{c^2}$

b. $\dfrac{c}{s\frac{c}{s}} = 1$

e. $\dfrac{s+\frac{s}{c}}{c+1} = \frac{s}{c}$

6. Write each fraction with a common denominator. (Scott O'Neill Rule)

a. $\frac{c}{s}$, $\frac{s}{c}$

b. $\frac{1}{c+1}$, $\frac{1}{c-1}$

c. $\frac{1}{c+s}$, $\frac{1}{c-s}$

d. $\frac{s}{1-c}$, $\frac{1+c}{s}$

e. $\frac{c}{c+s}$, $\frac{c-s}{c}$

Answer Key

a. $\frac{c^2}{sc}$, $\frac{s^2}{sc}$

b. $\frac{c-1}{(c+1)(c-1)}$, $\frac{c+1}{(c+1)(c-1)}$

c. $\frac{c-s}{(c+s)(c-s)}$, $\frac{c+s}{(c+s)(c-s)}$

d. $\frac{s^2}{s(1-c)}$, $\frac{1-c^2}{s(1-c)}$

e. $\frac{c^2}{c(c+s)}$, $\frac{c^2-s^2}{c(c+s)}$

LG 14 Worksheet B (Trig Proofs)

For each of the following, write an algebraic proof.

Identities will always have the following two properties:

1) If you graph the left and right sides, you will obtain exactly the same graph.

2) If you plug in the same angle for x on both sides, you will obtain exactly the same number.

1) Prove: $\cot x \tan x = 1$

2) Prove: $\csc x \cos x = \cot x$

3) Prove: $\dfrac{\sin x}{\tan x} = \cos x$

4) Prove: $\dfrac{1}{\cot x \cos x \tan x} = \sec x$

5) Prove: $\frac{\tan x}{\csc x} = \frac{\sin^2 x}{\cos x}$

6) Prove: $\frac{\tan x}{\sec x} = \sin x$

7) Prove: $\frac{\cos^2 x}{\cot x} = \sin x \cos x$

8) Prove: $\frac{\sec x \csc x}{\cot x} = \sec^2 x$

9) Prove: $\frac{\sec x \csc x}{\csc^2 x} = \tan x$

10) Prove: $\frac{\tan^2 x \cos x}{2 \sec x} = \frac{1}{2} \sin^2 x$

Questions: ***For each of the following, write an algebraic proof.***

1) $\sec x - \sin x = \frac{1 - \sin x \cos x}{\cos x}$

2) $\sin x + \tan x \sin x = \frac{\sin x \cos x + \sin^2 x}{\cos x}$

3) $\sec^2 x + \cot x = \frac{\sin x + \cos^3 x}{\cos^2 x \sin x}$

4) $\csc^2 x - \tan x = \frac{\cos x - \sin^3 x}{\sin^2 x \cos x}$

5) $\csc x - \sec x = \frac{\cos x - \sin x}{\sin x \cos x}$

6) $\sec x - \tan x = \frac{1 - \sin x}{\cos x}$

7) $\cos x + \tan x = \dfrac{\cos^2 x + \sin x}{\cos x}$

8) $\cot x + \sin x = \dfrac{\cos x + \sin^2 x}{\sin x}$

9) $1 + \tan x = \dfrac{\cos x + \sin x}{\cos x}$

10) $\csc x + 1 = \dfrac{1 + \sin x}{\sin x}$

Questions: *Use the special identities to do each of the following proofs.*

1) $\sec x - \tan x \sin x = \cos x$

2) $\cos x + \tan x \sin x = \sec x$

3) $\tan x + \cot x = \sec x \csc x$

4) $1 + \tan^2 x = \sec^2 x$

5) $\sec x - \cos x = \tan x \sin x$

6) $\sin x + \cot x \cos x = \csc x$

7) $\sec^2 x - 1 = \sin^2 x \sec^2 x$

8) $1 - \csc^2 x = -\cot^2 x$

9) $\csc x - \sin x = \cos x \cot x$

10) $1 - \sec^2 x = -\tan^2 x$

Questions: *Prove each of the following:*

1) $\frac{\sec x}{\cot x+\tan x}=\sin x$

2) $\frac{\sin x+\tan x}{\cos x+1}=\tan x$

3) $\frac{\cos x-\csc x}{\sin x-\sec x}=\cot x$

4) $\frac{\sin x+\cos x}{\sec x+\csc x}=\sin x\cos x$

5) $\frac{\tan x-\sin x}{\tan x\sin x}=\frac{1-\cos x}{\sin x}$

6) $\frac{1+\cos x}{\tan x+\sin x}=\cot x$

7) $\frac{1+\tan^2 x}{1+\cot^2 x}=\tan^2 x$

8) $\frac{1}{\sec^2 x}+\frac{1}{\csc^2 x}=1$

9) $\frac{1+\tan x}{1+\cot x}=\tan x$

10) $\frac{\cos x}{\sec x-1}+\frac{\cos x}{\sec x+1}=2\cot^2 x$

11) $\frac{\tan x}{1+\tan x}=\frac{\sin x}{\sin x+\cos x}$

12) $\frac{\sin^2 x}{1-\sin x}+\frac{\sin^2 x}{1+\sin x}=2\tan^2 x$

1) $\frac{3\tan x}{1+\tan^2 x} = 3\sin x\cos x$

2) $\frac{1}{1+\cot^2 x} = \sin^2 x$

3) $\sec^2 x - \cos^2 x - \sin^2 x = \tan^2 x$

4) $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$

5) $(1+\sin x)^2 + \cos^2 x = 2(1+\sin x)$

6) $\sin^4 x - \cos^4 x = 2\sin^2 x - 1$

Prove each of the following identities:

1) $\frac{\cos x}{1-\sin x}=\frac{1+\sin x}{\cos x}$ **2)** $\frac{1}{1-\sin x}=\frac{1+\sin x}{\cos^2 x}$ **3)** $\frac{1-\cos x}{\sin x}=\frac{\sin x}{1+\cos x}$ **4)** $\frac{1-\sin x}{\cos x}=\frac{\cos x}{1+\sin x}$

Double-Angle Identities

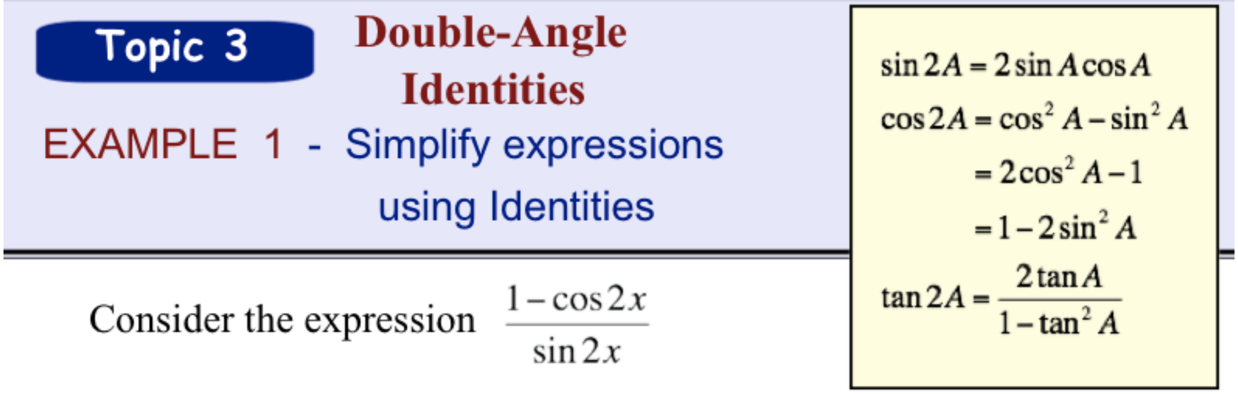
sin 2A =

cos 2A =

=

=

tan 2A =



Topic 3

Double-Angle Identities

EXAMPLE 1 - Simplify expressions using Identities

$\sin 2A = 2\sin A\cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$= 2\cos^2 A - 1$

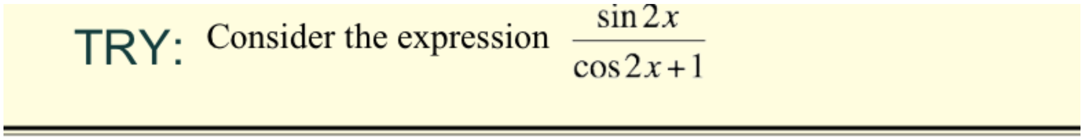
$= 1 - 2\sin^2 A$

$\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

Consider the expression $\dfrac{1-\cos 2x}{\sin 2x}$

a) What are the permissible values for the expression?

b) Simplify the expression to one of the three primary trigonometric functions.



TRY: Consider the expression $\dfrac{\sin 2x}{\cos 2x + 1}$

a) What are the permissible values for the expression?

b) Simplify the expression to one of the three primary trigonometric functions.

EXAMPLE 2 - Prove an Identity Using Double-Angle Identities

1. $\frac{1-\cos 2x}{\sin 2x}$ =	$\tan x$

2. $\frac{\cos 2x-\cos x}{\sin 2x+\sin x}$ =	$\cot x-\csc x$

TRY: Prove the following identity

$\frac{\sin 2x}{\cos 2x+1}$ =	$\tan x$

Try: $\frac{\sin^2 x\cos x+\cos^3 x}{2\sin x+1}$ =	$\frac{\sin 2x-\cos x}{4\sin^2 x-1}$

LEARNING GUIDE 15

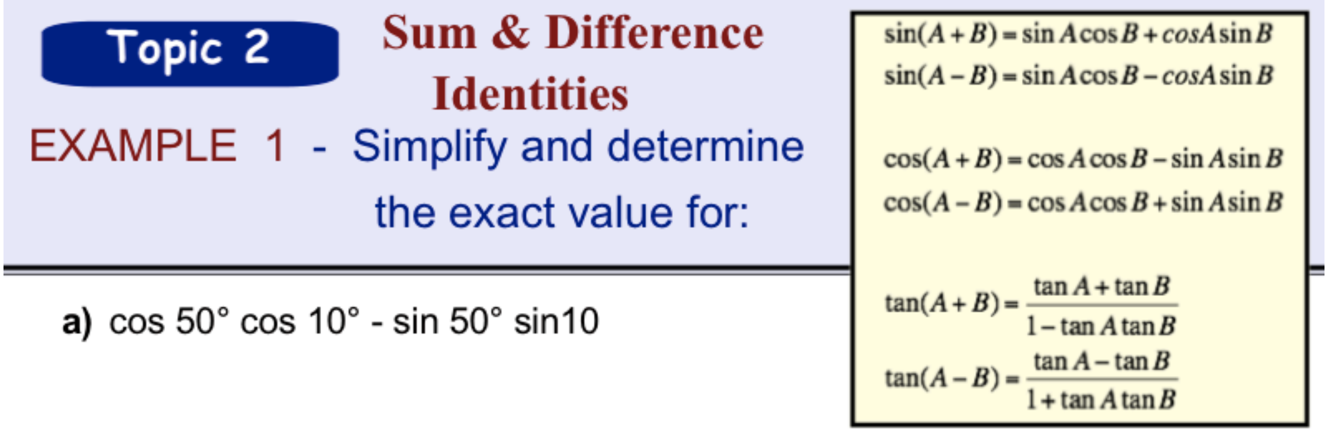
Part 2

Sum and Difference Identities

$$\sin(A+B)=\sin A\cos B+cosA\sin B$$
$$\sin(A-B)=\sin A\cos B-cosA\sin B$$

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$
$$\cos(A-B)=\cos A\cos B+\sin A\sin B$$

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$
$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$



Topic 2

Sum & Difference Identities

EXAMPLE 1 - Simplify and determine the exact value for:

$$\sin(A+B)=\sin A\cos B+cosA\sin B$$
$$\sin(A-B)=\sin A\cos B-cosA\sin B$$

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$
$$\cos(A-B)=\cos A\cos B+\sin A\sin B$$

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$
$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

a) cos 50° cos 10° - sin 50° sin10

b) $\sin\frac{5\pi}{12}\cos\frac{\pi}{12}+\cos\frac{5\pi}{12}\sin\frac{\pi}{12}$

EXAMPLE 1b - Simplify and determine the exact value for:

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

$\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

$c)\ \dfrac{\tan 70^\circ + \tan 80^\circ}{1 - \tan 70^\circ \tan 80^\circ}$

$d)\ \cos 80^\circ \cos 40^\circ - \sin 80^\circ \sin 40^\circ$

TRY:

1. Simplify and then give an exact value for the expression.

$$\cos\frac{\pi}{3}\cos\frac{\pi}{12} + \sin\frac{\pi}{3}\sin\frac{\pi}{12}$$

2. Simplify the expression to a single primary trigonometric function.

$$\sin 2x\cos x - \cos 2x\sin x$$

Sum & Difference Identities

EXAMPLE 2 - Determine Exact Trigonometric Values for angles

$$\sin(A+B)=\sin A\cos B+cosA\sin B$$
$$\sin(A-B)=\sin A\cos B-cosA\sin B$$

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$
$$\cos(A-B)=\cos A\cos B+\sin A\sin B$$

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$
$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

a) sin 15° b) cos 105° c) tan 195°

EXAMPLE 2b - Determine Exact Trigonometric Values for angles

d) $\sin\frac{-\pi}{12}$ e) $\cos\frac{7\pi}{6}$

TRY: Find the Exact value using Sum/Difference Identities.

1. $Sin75°$ **2.** $Cos\frac{\pi}{12}$ **3.** $\tan\frac{11\pi}{12}$

EXAMPLE 3 - Use the appropriate formula to simplify the following:

a) $\text{Cos } 3\theta \text{ Cos } \theta + \text{Sin } 3\theta \text{Sin } \theta =$

b) $\text{Cos } 7x \text{ Cos } 2x - \text{Sin } 7x \text{ Sin } 2x =$

EXAMPLE 4 - You can also use the Sum & Difference Identities to expand and simplify:

$$Sin\left(\frac{\pi}{2} - \theta\right) =$$

EXAMPLE 4b - You can also use these identities in proofs. Prove the following:

1. $Cos\left(\frac{\pi}{2} - \theta\right)$ =	$Sin\ \theta$

TRY: Prove the following identity:

12. $Sin\,(\pi - \theta)$ =	$Sin\theta$

EXAMPLE 5 - Sometimes you can combine the techniques you learned in the last section to simplify an expression:

Given Sin θ = $\frac{4}{5}$, where θ is in Quadrant II, evaluate the expression

$$Sin\left(\theta + \frac{\pi}{6}\right) =$$

TRY: *Given* $Cos\theta = \frac{-5}{13}$, *where* θ *is in Quadrant III*,

evaluate the expression $Cos\left(\frac{\pi}{2}-\theta\right) =$

LG 14 Worksheet C (Trig Proofs)

Find the exact value by expanding each of the following:

1) $\sin(45^\circ + 60^\circ)$

5) $\sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$

2) $\cos(45^\circ - 30^\circ)$

6) $\sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$

8) $\cos\left(\frac{\pi}{2} + \frac{\pi}{3}\right)$

For each of the following, express as a single trigonometric expression and solve using the unit circle.

1) $\cos 60^\circ \cos 15^\circ + \sin 60^\circ \sin 15^\circ$

2) $\cos^2 \frac{\pi}{6} - \sin^2 \frac{\pi}{6}$

6) $\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{3}\right)$

8) $\cos\left(\frac{2\pi}{3}\right)\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{2\pi}{3}\right)\sin\left(\frac{\pi}{2}\right)$

Find the exact value of each of the following:

Note that there are multiple ways of getting to the correct answer.

1) $\cos(-15°)$

2) $\sin(105°)$

4) $\sin(-\frac{5\pi}{12})$

Prove each of the following:

1) $\cos(\frac{3\pi}{2} - x) = -\sin x$

5) $\sin(\frac{\pi}{2} - x) =$

3) $\cos(\frac{\pi}{2} + x) = -\sin x$

4) $\cos(\pi - x) =$

$\sin x = -\frac{1}{2}$ (Quadrant III)

1)

$\tan y = \frac{3}{4}$ (Quadrant I)

$\boxed{\text{Find } \cos(x-y)}$

$\sec x = \frac{7}{5}$ (Quadrant I)

3)

$\cot y = -\frac{3}{4}$ (Quadrant II)

$\boxed{\text{Find } \sin(x-y)}$

In the following examples, the double-angle identities will be used in completing proofs.

1) $\cos 2x + \cos x = (2\cos x - 1)(\cos x + 1)$

2) $2\cos 2x - \sin x + 1 = -(4\sin x - 3)(\sin x + 1)$

3) $\cos 2x = 2\cos^2 x - 1$

4) $\cos 2x = 1 - 2\sin^2 x$

5) $\dfrac{1+\cos 2x}{\sin 2x}=\cot x$

6) $\dfrac{2}{1+\cos 2x}=\sec^2 x$

7) $\sin 2x=2\sin x\cos x$

8) $\cos 2x-1+2\sin x=2\sin x(1-\sin x)$

EXAMPLE 3 - Express as a Single Trigonometric function.

a) $4 Sin 60° Cos 60°$

b) $4 - 8\sin^2 4x$

TRY: - Express as a Single Trigonometric function.

a) $8\cos^2 2x - 4$

b) $\sin 3x \cos 3x$

TRY:

1. $sin^2 x + cos^2 x$
2. $cos^2 x - sin^2 x$
3. $10 sinx cosx$
4. $8 sinx cosx$
5. $2 sin3x cos3x$
6. $10 sin5x cos5x$
7. $2 - 4 sin^2 5x$
8. $6 cos^2 4x - 3$

Answer Key

1. 1 2. $cos2x$ 3. $5sin2x$ 4. $4sin2x$ 5. $sin6x$ 6. $5sin10x$ 7. $2cos10x$ 8. $3cos8x$

PRE-CALCULUS 12

Seminar Notes

Learning Guides 16

TRIGONOMETRIC EQUATIONS

LEARNING GUIDE 16

Topic 1

Solving by Graphing Calculator

EXAMPLE 1 - Using your graphing calculator solve the

$$Sin\ x = 0.45$$

- **Write an expression that represents all the roots of the equation Sin x = 0.45**

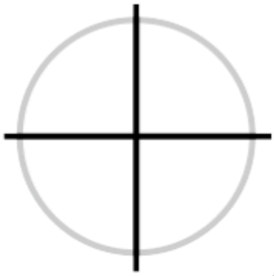
Topic 2

Finding Exact Value Solutions

Remember
CAST Rule

EXAMPLE 1 - Find the Exact and General Solutions

- Solve the equation: $Cos\ x = \frac{-1}{2}$ for $0 \le x < 2\pi$
  (Give exact solutions)



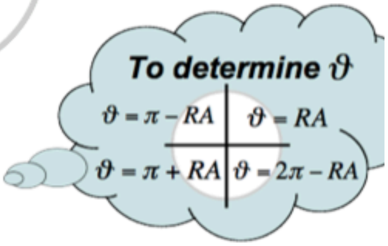
Steps to solve trigonometric functions:

1. Looking at the sign of the original equation and then draw the reference angle in the appropriate quadrants.

2. Ignore the negative sign if present.

3. Find the reference angle (**RA)** by using your a) **calculator** b) **unit circle** or, in this case, c) **special triangles.**

4. Determine the angles in standard position (ϑ) These are your solutions



What are the general solutions for this equation?

EXAMPLE 2 - Find the exact value solution for the following equation over the domain $0° \le \theta \le 360°$

Read carefully the domain

$$Sin\ \ x = \frac{-1}{2}$$

TRY:

- Solve the equation: $Tan\ \ x = \frac{-1}{\sqrt{3}}$ for $0 \le x < 2\pi$
  (Give exact solutions)

TRY:

Solve: $\text{Sec } x = \sqrt{2}$ for $0 \le x < 2\pi$
(Give exact solutions)

Hint: *rewrite the equation in terms of Cos x first*

Let's Practice Isolating $sinx =, \; cosx =, \; tanx =$

Find the exact value of A if $0 \le A < 2\pi$

a. $2SinA + 1 = 0$

b. $\sqrt{3} + 2SinA = 0$

c. $2TanA = 2$

d. $4SinA + 2 = 2SinA + 1$

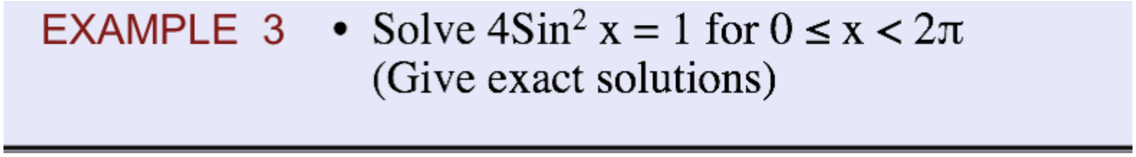
3. Solve to 2 decimal places, if $0 \le x < 2\pi$

a. $3Sinx + 2 = 0$

b. $5Cosx - 4 = 0$

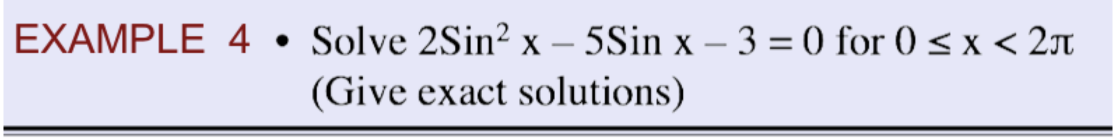
c. $4Tanx + 1 = 2Tanx + 5$

4. Why do the equations of SinA = 2 and CosA = -3 have no solutions, and TanA = 4 have solutions?



EXAMPLE 3 • Solve $4\text{Sin}^2 x = 1$ for $0 \le x < 2\pi$
(Give exact solutions)

Hint: ***solve using square root method***



EXAMPLE 4 • Solve $2\text{Sin}^2 x - 5\text{Sin } x - 3 = 0$ for $0 \le x < 2\pi$
(Give exact solutions)

Hint: ***solve quadratic equation***

Write the general solution

EXAMPLE 5 • Solve $\text{Cos}^2 x - \text{Cos } x = 0$ for $0 \le x < 2\pi$
(Give exact solutions)

Hint: *factor out Cos x first*

Write the general solution

Topic 3 Finding Solutions to the Nearest Hundredth

EXAMPLE 1 - Solve each equation for given domain $0 \le \theta \le 2\pi$.
Give solutions to the nearest hundredth.

Beware of Rejections

a) $\sin x = \frac{1}{3}$

b) $\cos x = -0.625$

c) $4\cos^2 x - 7\cos x = 2$

LG 16 Worksheet A (Trig Equations)

1. **Find the exact value of A if 0 ≤ A < 360°**

a. $\text{SinA} = \frac{1}{2}$

b. SecA = -2

c. CotA = -1

d. $\text{TanA} = \sqrt{3}$

e. CosA = 0

f. SinA = 0

2. Find the exact value of A if $0 \le A < 2\pi$

a. CosA = $\frac{-1}{2}$

b. CscA = $-\sqrt{2}$

c. CotA = 1

d. SecA = 1

e. CscA = -2

f. TanA = 0

3. Find A to 1 decimal place if $0 \le A < 360°$

a. CosA = -0.7819

b. CscA = -2.3451

c. SecA = 6.5789

d. CotA = 0.2134

4. Find A to 2 decimal places if $0 \le A < 2\pi$

a. TanA = 0.6781

b. SecA = -2.4567

c. CscA = 8

d. CotA = -0.9145

5.Find the exact value of A if 0 ≤ A < 360°

a. $\text{Sin}^2 A = 1$

b. $\text{Cos}^2 A = \frac{1}{4}$

c. $\text{Tan}^2 A = \frac{1}{3}$

d. $\text{Sec}^2 A = 2$

e. $\text{Csc}^2 A = \frac{4}{3}$

f. $\text{Cot}^2 A = \frac{1}{3}$

6. Find the exact value of A if 0 ≤ A < 2π

a. $\text{Csc}^2 A = 4$

b. $\text{Sec}^2 A = \frac{4}{3}$

c. $\text{Cot}^2 A = 1$

d. $\text{Cos}^2 A = \frac{1}{2}$

e. $\text{Sin}^2 A = \frac{3}{4}$

f. $\text{Tan}^2 A = 3$

7. Solve each of the following equations algebraically for x, 0 ≤ x < 2π. Give exact values where possible (otherwise to 2 dec. places). Also, solve over the set of real numbers (give the general solution).

$a)\ cos^2x - cosx - 2 = 0$

$b)\ sin^2x - 4sinx + 3 = 0$

$c)\ 2cos^2x - cosx = 1$

$d)\ 2sin^2x + 3sinx = 2$

$e)\ 6cos^2x = cosx + 1$

$f)\ tan^2x + 3tanx = -2$

Answer Key

1. a. $30^\circ, 150^\circ$ b. $120^\circ, 240^\circ$ c. $135^\circ, 315^\circ$

d. $60^\circ, 240^\circ$ e. $90^\circ, 270^\circ$ f. $0^\circ, 180^\circ$

2. a. $\frac{2\pi}{3}, \frac{4\pi}{3}$ b. $\frac{5\pi}{4}, \frac{7\pi}{4}$ c. $\frac{\pi}{4}, \frac{5\pi}{4}$

d. 0 e. $\frac{7\pi}{6}, \frac{11\pi}{6}$ f. $0, \pi$

3. a. $141.4^\circ, 218.6^\circ$ b. $205.2^\circ, 334.8^\circ$

c. $81.3^\circ, 278.7^\circ$ d. $78.0^\circ, 258.0^\circ$

4. a. $0.60, 3.74$ b. $1.99, 4.29$

c. $0.13, 3.02$ d. $2.31, 5.45$

5. a. $90^\circ, 270^\circ$ b. $60^\circ, 120^\circ$
$240^\circ, 300^\circ$

c. $30^\circ, 150^\circ$
$210^\circ, 330^\circ$ d. $45^\circ, 135^\circ$
$225^\circ, 315^\circ$

e. $60^\circ, 120^\circ$
$240^\circ, 300^\circ$ f. $60^\circ, 120^\circ$
$240^\circ, 300^\circ$

6. a. $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ b. $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

c. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ d. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

e. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ f. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

7. a) $\boldsymbol{\pi}\ (+\boldsymbol{2\pi n, n\epsilon I})$ b) $\boldsymbol{\frac{\pi}{2}}\ (+\boldsymbol{2\pi n, n\epsilon I})$ c)) $0, \boldsymbol{\frac{2\pi}{3}, \frac{4\pi}{3}}\ (+\boldsymbol{2\pi n, n\epsilon I})$

d) $\boldsymbol{\frac{\pi}{6}, \frac{5\pi}{6}}\ (+\boldsymbol{2\pi n, n\epsilon I})$ e) $1.91, 4.37, \boldsymbol{\frac{\pi}{3}, \frac{5\pi}{3}}\ (+\boldsymbol{2\pi n, n\epsilon I})$ f) $2.03, 5.18, \boldsymbol{\frac{3\pi}{4}, \frac{7\pi}{4}}\ (+\boldsymbol{2\pi n, n\epsilon I})$

Topic 4 Solving Equations with Specific Domains

Watch Out!!

- Be sure to carefully read the question and what it is asking you to answer from.

EXAMPLE 1. - $Sin\ \ x = 0 \quad for\ \ 0 \le x < 2\pi$

EXAMPLE 2. - $Sin\ \ x = 0 \quad for\ \ 0 \le x \le 2\pi$

EXAMPLE 3 - Solve the following equation in the specific domain.

$$\cos x = \frac{1}{\sqrt{2}} \quad for \; -\pi \le x < \pi$$

TRY: Determine the exact roots for each trigonometric equation in the specific domain.

$a)\ \sin\theta = \frac{1}{\sqrt{2}},\ -180° \le \theta < 360°$

$b)\ \sqrt{3}\sec\theta + 2 = 0,\ -\pi \le \theta < 3\pi$

Topic 5 Solving Equations With an Identity Substitution

EXAMPLE 1 - Solve the equation $\cos^2 x = \cot x \sin x$ algebraically in the domain $0^0 \le x < 360^0$.

TRY: Solve the equation $\sin^2 x = \frac{1}{2}\tan x \cos x$ algebraically over the domain $0^0 \le x < 360^0$.

EXAMPLE 2 - **Algebraically solve 2 sin x = 7 - 3 csc x , and then give general solutions in radians.**

TRY: Algebraically solve 3 cos x + 2 = 5 sec x , and then give general solutions in radians.

Topic 6

Solve by Factoring

EXAMPLE 1 - Solve exactly: $2SinxTanx - Tanx = 0 \quad for\ 0 \le x < 2\pi$

EXAMPLE 2 - **Solve the equation $\sin 2x = \sqrt{2} \cos x$ algebraically, and then give the general solution expressed in radians.**

TRY: **Algebraically solve $\cos 2x = \cos x$.**
Give general solutions expressed in radians.

LG 16 Worksheet B (Trig Equations)

1. **Solve the equation to the right over each of the given domains:** $2sin^2x - sinx - 1 = 0$

$\boldsymbol{a})\ \ 0 \le x < 2\pi$

$\boldsymbol{b})\ \ \pi \le x < 2\pi$

$\boldsymbol{c})\ \ 0 \le x < \frac{\pi}{2}$

$\boldsymbol{d})\ \ \frac{\pi}{2} \le x < \frac{3\pi}{2}$

$\boldsymbol{e})\ \ \frac{\pi}{2} \le x \le 2\pi$

$\boldsymbol{f})\ -\pi \le x < 0$

2. **Solve each of the following equations algebraically for x, 0 ≤ x < 2π. Give exact values where possible (otherwise to 2 dec. places).**

$a)\ \ 2sinxtanx - tanx = 0$

$b)\ \ tanx - 2cosxtanx = 0$

$c)\ \ cosxtanx - 3tanx = 0$

$d)\ \ 3costanx - tanx = 0$

$e)\ \ tan^2x = tanx + 2$

$f)\ \ sec^2x - 2secx = 3$

3. Use *identities* to solve each of the following equations algebraically for x, 0 ≤ x < 2π. Give exact values where possible (otherwise to 2 dec. places).

$a)\ \ sin2x - sinx = 0$

$b)\ \ cosx - sin2x = 0$

$c)\ \ cosx = cos2x$

$d)\ \ sinx - cos2x = 0$

Answer Key

1. a) $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **b)** $\frac{7\pi}{6}, \frac{11\pi}{6}$ **c)** *no solution* **d)** $\frac{\pi}{2}, \frac{7\pi}{6}$ **e)** $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **f)** $\frac{-5\pi}{6}, \frac{-\pi}{6}$

2. a) $0, \pi, \frac{\pi}{6}, \frac{5\pi}{6}$ **b)** $0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}$ **c)** $0, \pi$ **d)** $0, \pi, 1.23, 5.05$ **e)** $1.11, 4.25, \frac{3\pi}{4}, \frac{7\pi}{4}$ **f)** $\pi, 1.23, 5.05$

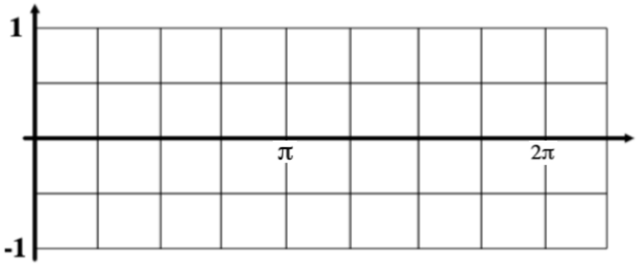
3. a) $0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}$ **b)** $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **c)** $0, \frac{2\pi}{3}, \frac{4\pi}{3}$ **d)** $\frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$

Topic 7

Solving Multiple Angles

EXAMPLE 1 Solve $Sin2x = \frac{1}{2}$; over [0, 2π]

Let's look at it visually first.



Now algebraically.

EXAMPLE 2 Solve $Cos3x = -\frac{\sqrt{3}}{2}$ over [0, 2π].

You Try:

a) Solve $Sin2x = \frac{1}{\sqrt{2}}$ over [0, 2π]

b) Solve $Cos3x = 0$ over [0, 2π].

c) $\cos 2x = -\frac{1}{\sqrt{2}}$,

a) if the domain is $0 \le x < 2\pi$

b) if the domain is $-\pi \le x < \pi$

PRE-CALCULUS 12

Review Questions
for
Final Exam

***SEE YOUR TEACHER FOR ANSWER KEY

Learning Guides 1/2:

1. If (-3,-1) is a point on the graph of $y = f(x)$, find a point on the graph of each of the following:

a) $y = f(-x)$

b) $y = -f(x)$

c) $y = -f(-x)$

d) $y + 4 = f(x + 1)$

e) $x = f(y)$ and give **three** other ways this question could have been asked

---

2. If (c, d) is a point on the graph of $y = f(x)$, find a point on the graph of each of the following:

a) $y = 3f(x) - 4$

b) $y = f(-3x) - 1$

c) $y = 4f(6x + 12) - 7$

d) $y = -2f(8 - 4x) - 6$

e) $y - 4 = -3f(2x - 10)$

f) $2y + 6 = f(8 - 8x)$

3. **If f(x)** $f(x) = 5x^3 - 6x^2 + 2$ **find the equation of each of the following:**

a) $y = -f(x)$

b) **the reflection of** $f(x)$ **in the y-axis**

c) $y = f(3x)$

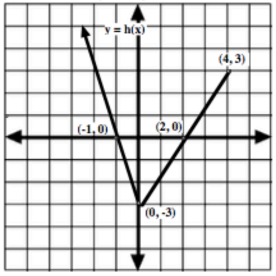
4. **Given** $y = f(x)$ **find** $y = f^{-1}(x)$ **(or** $x = f(y)$**, or the reflection in the line** $y = x$ **or the inverse):**

a) $f(x) = \dfrac{-3x}{5+x}$ **b)** $f(x) = \sqrt{3x-1} + 2$ **c)** $f(x) = 3(x+2)^2 - 4$

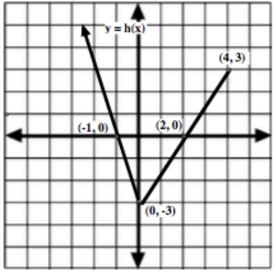
5. **Describe how the graph of** $y = -2f(4x - 12) - 5$ **is related to the graph of** $y = f(x)$**.**

6. **Given the graph of** $y = f(x)$ **below sketch each of the following graphs:**

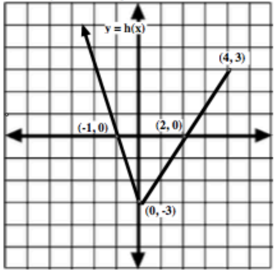
a) $y = f^{-1}(x)$



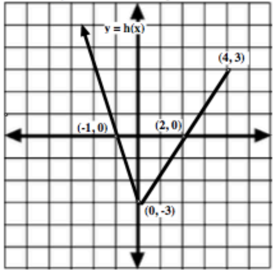
b) $y + 2 = f(x + 3)$



c) $y = -3f(2x - 4) + 5$



d) $y + 1 = -2f(4 - x)$



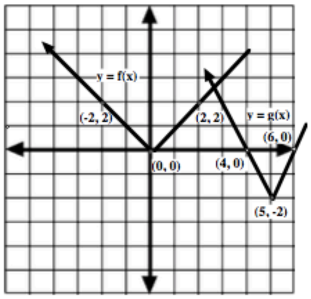
7. **If** $f(x) = x^2 + 1$ **find the equation of the transformed curve** $y = 3f(x-1) - 4$**.**

---

8. **If the domain of** $y = f(x)$ **is** $x \le -3$ **and the range is** $y > 5$ **find the domain and range of** $y = 3f(x+4)$**.**

---

9. **The graph of** $y = f(x)$ **has been transformed to the graph of** $y = g(x)$**. Write the equation for the graph of** $g(x)$ **in terms of** $f(x)$**.**



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10. **The x-intercepts of the graph of** $y = f(x)$ **are (-4, 0) & (6, 0) and the y-intercept is (0, -8). If** $h(x) = -3f(2x)$**, determine the x & y-intercepts of** $h(x)$**.**

11. **a)** **If** $f(x) = 4x - 2$**, find is** $f^{-1}(8)$**.**

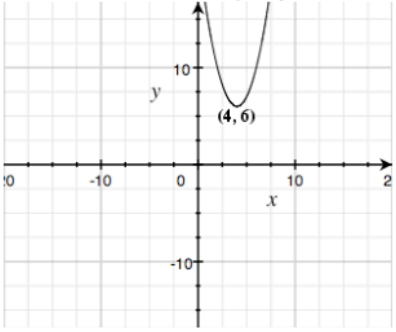
b) **If (10, 8) is on the graph of** $y = f(x)$**, what point must be on the graph of each of the following:**

i) $y = f^{-1}(x+2)$

ii) $y = 2f^{-1}(x) + 3$

iii) $y = -f^{-1}(-x) + 1$

12. **Given the graph of** $y = f(x)$**, restrict its domain so that** $y = f^{-1}(x)$ **is a function.**



13. **If the following points lie on the graph of** $y = f(x)$**, which are invariant points for is** $y = f(x)$ **and** $y = f^{-1}(x)$**?** **(4, 3)** **(2, -2)** **(5, 5)** **(9, 9)** **(9, -9)**

Learning Guide 3

1. Match each function to its corresponding graph.

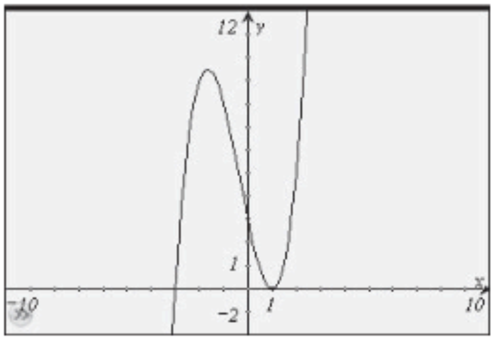
a) $g(x) = -x^4 + 10x^2 + 5x - 4$

b) $f(x) = x^3 + x^2 - 5x + 3$

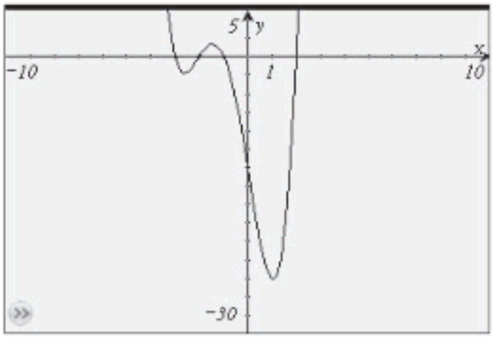
c) $p(x) = -2x^5 + 5x^3 - x$

d) $h(x) = x^4 + 4x^3 - x^2 - 16x - 12$

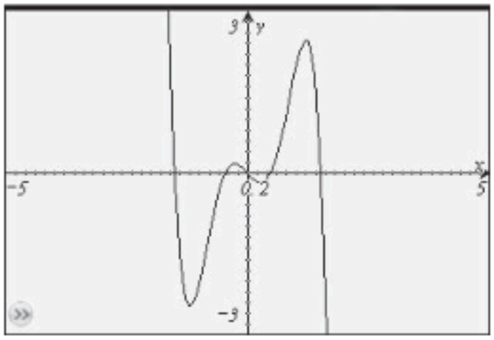
A



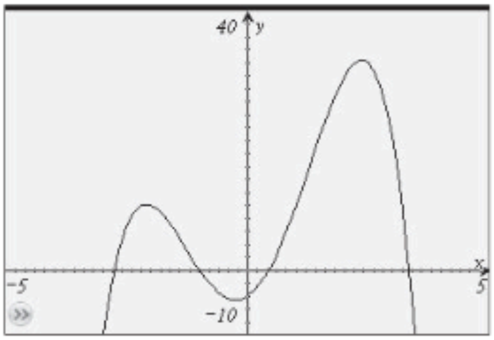
B



C



D



2.

For the function $y = (x-1)(x+3)(x-4)$ determine each of the following:

a) x-intercept(s)

b) y-intercept(s)

c) the degree

d) end behaviour of the graph

e) the interval(s) where the function is positive

f) the interval(s) where the function is negative

3. Use synthetic division to determine the remainder for $x^3 + 7x^2 + 3x + 4 \quad \div \; x + 2$.

---

4. Use the remainder theorem to determine the remainder when $P(x) = x^3 - 10x + 6$ is divided by $x + 4$.

---

5. Determine the value of *k* if the remainder is 3 for $(kx^3 + 3x + 1) \div (x + 2)$.

---

6 a)

For what value of k will the polynomial $2x^4 - 8x^2 + kx - 20$ **have a remainder of 49 when it is divided by** $x - 3$**.**

---

6 b)

Determine the value of k **so that** $x + 3$ **is a factor of** $x^3 + 4x^2 - 2kx + 3$**.**

7.

Determine which of the following binomials are factors of the polynomial $P(x) = x^3 + 2x^2 - 5x - 6$.

$x - 1, x + 1, x - 2, x + 2, x - 3, x + 3, x - 6, x + 6$

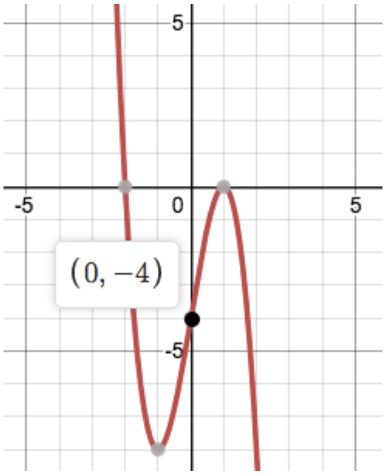
8.

Factor $3x^3 + 2x^2 - 7x + 2$ fully. You must show your synthetic or long division to receive full marks.

---

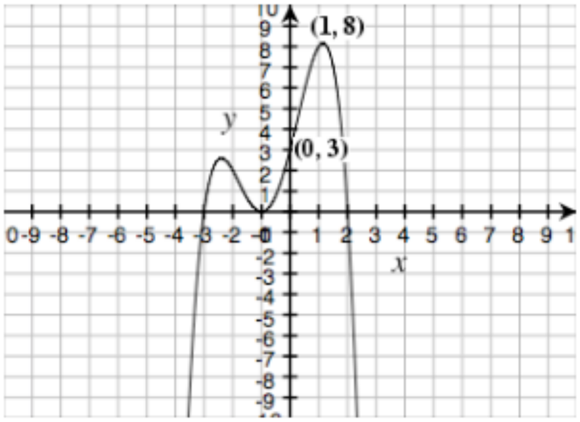
9. Factor fully $x^3 - 4x^2 - 11x + 30$.

10a) Determine the equation of the function shown on the graph below.



10b)

Use the graph of the given polynomial function to write its equation.



11. Determine the equation of the function that the zeros are -2, -1, and 3 (multiplicity 2) and passes through the point (2, 24).

12.

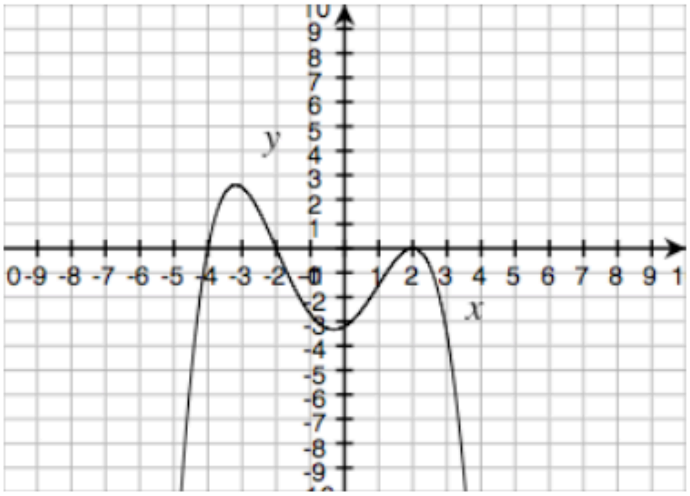
The zeros of a quartic function are 1, -2, and -4 (*multiplicity 2*). Determine the equation of the function that has these zeros and passes through the point (-3, 20)

13.

If the function $h(x)$ results from performing the following transformations on $f(x) = x^3$, find the equation for $h(x)$:
vertically stretched (*EXPANDED*) by a factor of 4,
reflected in the y-axis,
translated 8 units right,
4 units down.

14.

For the polynomial graphed below, determine its least possible degree and the sign of the leading coefficient.



Learning Guide 4/5

1. Solve for x:

a) $16^{2x-1} = 4^{3x+5}$

b) $\left(\frac{1}{9}\right)^{2x-3} = 81^{3x+1}$

---

2. Bacteria, with an initial population of 200, grows to 6400 in 12 hours. What is the doubling period of this bacterium?

---

3. A 160 g sample of material decays to 8 g in 22 minutes. What is the half-life of this material? Solve algebraically using logarithms. Give your answer accurate to 3 decimal places.

---

4 a) **Write** $k = m^{\frac{p}{q}}$ **in logarithmic form**

b) **Write** $x = \log_y b$ **in exponential form**

c) **Write an equivalent expression for** $\log_9 7$ **in base 6.**

d) **Solve for x:**

i) $\log_{\frac{1}{3}} x = 2$

ii) $\log_x 64 = 2$

iii) $\log_{\sqrt{3}} 27 = x$

iv) $5^{2x} - 6(5)^x + 5 = 0$

v) $2^x + 2^{x-3} = 18$

5. **Write the following expression as a single logarithm:** $\frac{1}{2}\log B - 3\log C - \frac{1}{4}\log D + 5\log E$

---

6. **Solve for x exactly:**

a) $4 = 6^{2-x}$

b) $5^{x+1} = 8^{2x-3}$

---

7. **Solve for x:**

a) $\log_3(x-6) + \log_3 x = 3$

b) $\log_5(1-x) + \log_5(-x) = \log_5 6$

c) $2\log_2(x+1) - \log_2(3-x) = 1$

8. **Given the function** $g(x) = 2(3)^{x+3} - 4$**, find the:**

a) y-intercept

b) equation of the horizontal asymptote

c) domain & range

---

9. a) If $x = \log 3$ **and** $y = \log 2$ **, find an expression for log 36.**

b) If $x = \log_2 3$**, find an expression for** $\log_2 8\sqrt{3}$ **in terms of x.**

c) If $x = \log_2 5$**, find an expression for** $\log_2 \sqrt[4]{25^3}$ **in terms of x.**

---

10. How much more powerful is an earthquake that measures 8.9 on the Richter Scale compared to one that measures 7.6?

---

11. A 200 g sample of radioactive polonium-210 has a half-life of 138 days.

a) Determine an equation to model this situation of the mass of polonium that remains after *t* days.

b) Determine the mass that remains after 5 years?

c) How long will it take for this 200 g sample to decay to 12.5 g?

12. Find the inverse of each of the following:

a) $y = \log_5(x-2)$

b) $y = 5^{2x+3}$

13. a) Write the following expression in terms of the individual logarithms of m, x, y and z.

$$\log_5 \frac{x^5 y^2}{m^3 \sqrt{z}}$$

b) Write the following as a single logarithm in simplest form. $2\log x + 3\log\sqrt{x} - \log x^3$ **State any restrictions.**

14. Simplify:

a) $\log_{b^m} b^k$

b) $\log_c c^k$

c) $\log_k k^2\sqrt{k}$

15. a) Evaluate: $\log_2\left(\log_5 \sqrt{5}\right)$

b) Solve for x. $\log_x\left(\log_3 \sqrt{27}\right) = \frac{1}{5}$

Learning Guide 6

1. Find the sum of the geometric series.

a) $S_7 \ \ of \ \ 4+8+16+32+\cdots$

b) $S_\infty \ \ of \ \ 1-6+36-216+\cdots$

c) $\frac{3}{25}+ \_\_ + \_\_ + \_\_ + \_\_ +375$

d) $S_6 =$ ___ given $t_2 = 1$ and $t_5 = \frac{-1}{27}$

---

2. A geometric series has a sum of 1365. Each term increases by a factor of 4. If there are 6 terms, find the value of the first term.

---

3. Rewrite each series as a sum.

a) $$\sum_{m=1}^{6} \frac{1}{5^m}$$

b) $$\sum_{k=4}^{8} \frac{k}{k+1}$$

---

4. Evaluate each series.

a) $$\sum_{m=1}^{5} \frac{7}{m}$$

b) $$\sum_{n=0}^{5} n(n-2)$$

---

5. Rewrite each series using sigma notation.

a) $4+2+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$

b) $0+3+6+9$

c) $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}$

d) $\frac{1}{5}+\frac{1}{3}+1-$

Learning Guide 7/8

1.

If $f(x)=\dfrac{1}{(x-1)}$ and $g(x)=\sqrt{x}$ determine the equation of the combined function $(f+g)(x)$ and state its domain and range.

2.

Given $g(x)=\dfrac{1}{(x+4)}$ and $h(x)=\dfrac{1}{(x^2-16)}$ determine the equation of the combined function $f(x)=\dfrac{g(x)}{h(x)}$ and state its domain and range.

3.

Given the functions $f(x)=\dfrac{1}{x}$ and $g(x)=\dfrac{1}{(x+1)}$ determine the equation of the function (*hx*) = (*fg*)(*x*), **and state the domain and range.**

4.

Given the functions $h(x)=x^2+x-6$ and $f(x)=2x+6$ determine the equation of the function $g(x)=\left(\dfrac{f}{h}\right)(x)$ and state the domain and range of $g(x)$.

5.

Given $f(x) = x^2$ **and** $g(x) = x + 1$ **find each of the following:**

a) $(f - g)(-2)$ **b)** $\left(\frac{g}{f}\right)(-3)$ **c)** $(f \circ g)(4)$ **d)** $g(f(4))$ **e)** $(g \circ g)(5)$

6.

If $f(x) = \frac{-2}{x}$ **and** $g(x) = -\sqrt{x}$ **determine** $y = f(g(x))$ **and state the domain and range of y.**

7.

If $f(x) = 2x^2$ **and** $g(x) = 4x$ **determine the following and state any restrictions:**

a) $g(f(-2))$ **b)** $(f \circ g)(x)$ **c)** $g(f(x))$

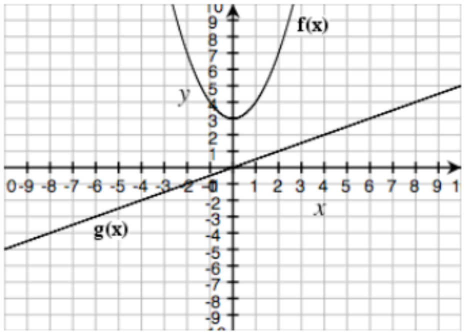
8.

For each of the following pairs of functions determine $g(f(x))$ **and state its domain and range.**

a) $f(x) = 3 - x$ **and** $g(x) = |x + 3|$ **b)** $f(x) = x^4$ **and** $g(x) = \sqrt{x}$

9.

The graphs of two functions is shown below. Which of the following statements are true for $x \in R$?



a) $g(x) - f(x) < 0$ **b)** $\frac{f(x)}{g(x)} > 1, x > 2$ **c)** $f(x) < g(x)$ **d)** $g(x) + f(x) < 0$

Learning Guide 10/11

1. a) Convert 140° to radians exactly

b) Convert $\frac{7\pi}{10}$ to degrees exactly

c) Find one positive and one negative co-terminal angle for $\frac{8\pi}{7}$ and its reference angle.

d) Write an expression for all of the angles co-terminal with $\frac{-3\pi}{11}$.

e) Find all angles θ co-terminal with $\frac{-11\pi}{6}$, $-4\pi \le \theta < \pi$.

2. **Find the exact value of each of the following:**

a) $\sec\frac{-5\pi}{6}$ **b)** $\tan 630^\circ$ **c)** $\csc\frac{11\pi}{3}$

---

3. **Find the exact value of each of the following:**

a) $\tan^3\left(\frac{-5\pi}{4}\right)$ **b)** $\sec^3\left(\frac{7\pi}{6}\right)$

c) $\cot^5\left(\frac{7\pi}{2}\right)$ **d)** $4\sin\left(\frac{4\pi}{3}\right)+6\sec\left(\frac{3\pi}{4}\right)$

---

4. **a)** **if** $\sin A=\frac{2}{5}$, $0 \le A < 2\pi$ **find the exact value(s) of** $\sec A$**.**

b) **if** $\cos A=\frac{-1}{3}$ **and angle A is in QII, find the exact value(s) of** $\csc A$.

c) **if** $\sin A=\frac{-1}{4}$ **and** $TanA > 0$ **find the exact value(s) of** $\cot A$.

d) **if** $\csc A=-5$ **and** $\frac{-\pi}{2} \le A < \frac{\pi}{2}$, **find the exact value(s) of** $\tan A$.

---

5. **A circle has a radius of 25 m. If a sector has an arc length of 60 m find the sector angle to the nearest tenth of a radian.**

Learning Guide 12/13

1. If $y = -6\cos\left(3x - \frac{\pi}{4}\right) - 5$, find the :

a) amplitude

b) period, in radians and degrees

c) vertical displacement

d) phase shift

e) domain and range

f) the maximum and minimum values

---

2. If $y = \tan x$, $-2\pi \leq A < 2\pi$, find the:

a) period, in radians and degrees

b) y-intercept

c) x-intercepts

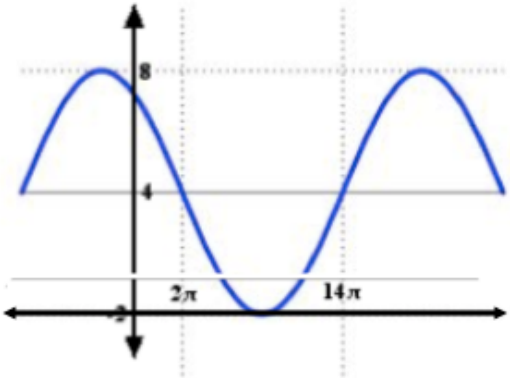
d) equation of one asymptote

e) domain and range

3. **Give the equation of a sine curve with amplitude of 3, period of 6, vertical displacement of 8 and phase shift of $\frac{2\pi}{7}$ to the left.**

---

4. **Given the sinusoidal curve graphed below, write its equation in the form** $y = a\sin b(x-c)+d$ **and** $y = a\cos b(x-c)+d$.



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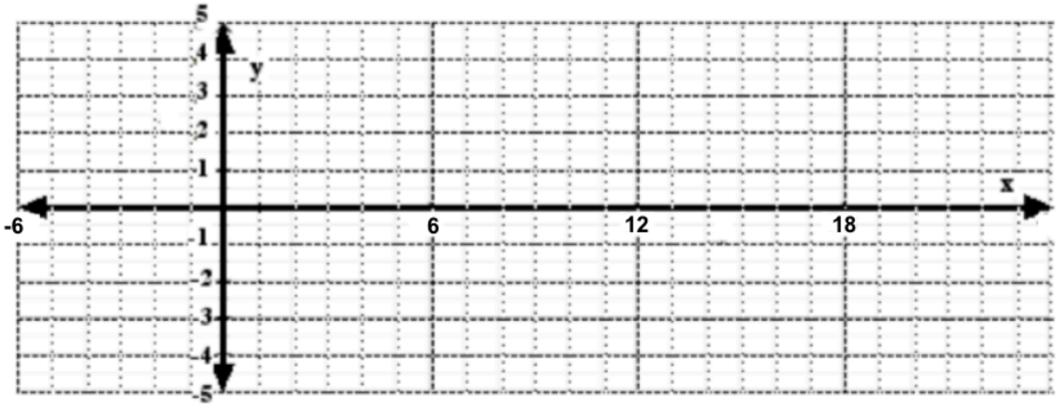
5. Graph $y = 2sin\left(\frac{\pi}{3}x - \frac{\pi}{3}\right) - 1$ for two cycles.



6. **Find a function in the form $y = A\sin B(x - C) + D$ if it has a maximum point of (2, 6) and the closest minimum point of (10, - 6).**

7.

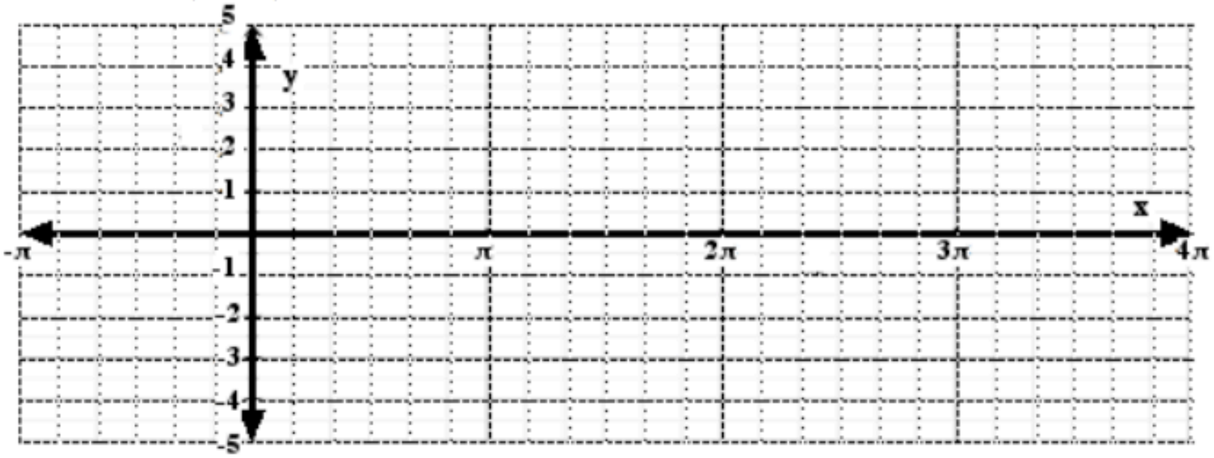
A Ferris wheel has a radius of 30 m and its center is 34 m above the ground. It rotates once every 80 seconds. Repete gets on the Ferris wheel at its lowest point and then the wheel begins to rotate.

a) **Determine a sinusoidal equation that gives Repete's height, h, above the ground as a function of the elapsed time, t, where h is in meters and t is in seconds**

b) **Determine the first time t, in seconds, when Repete will be 48 m above the ground**

---

8. **Graph $y = 2\sin\left(3x - \frac{\pi}{2}\right) - 1$ for two cycles.**



Learning Guide 14/15

1. Write as a single trigonometric function:

a) $6\sin 20^\circ \cos 20^\circ$

b) $1-2\cos^2\frac{5\pi}{4}$

c) $12\sin 2x\cos 2x$

d) $\cos^2\frac{5\pi}{7}-\sin^2\frac{5\pi}{7}$

e) $\sin 20x\cos 20x$

f) $8\cos^2 5x-4$

g) $6-12\cos^2 6x$

h) $5-10\sin^2 10x$

i) $\frac{\tan 2x-\tan 6x}{1+\tan 2x\tan 6x}$

j) $\frac{2\tan 10}{1-\tan^2 10}$

k) $\frac{\sin x+\tan x}{1+\cos x}$

l) $\frac{2-2\sin^2 A}{\cos 2A-1}$

---

2. If angles A and B are both in the second quadrant and $\sin A=\frac{4}{5}$ and $\cos B=\frac{-5}{13}$ find the exact value of each of the following:

a) $\sin(A-B)$

b) $\cos(A+B)$

2. If angles A and B are both in the second quadrant and $\sin A = \frac{4}{5}$ and $\cos B = \frac{-5}{13}$ find the exact value of each of the following:

c) $\sin 2A$

d) $\cos 2A$

---

3. Find the exact value of each of the following:

a) $\cos 105^\circ$

b) $\sin \frac{7\pi}{12}$

---

4. Prove each of the following identities:

a. $$\frac{\sin 2A}{1+\cos 2A} = \frac{1}{\cot A}$$

b) $$\frac{\csc x + 1}{\cot x} = \frac{\cot x}{\csc x - 1}$$

4. Prove each of the following identities:

c) $\frac{1-\cos 2x}{1-\sin^2 x} = \frac{2}{\cot^2 x}$

Learning Guide 16

1.

Solve each of the following equations, $0^\circ \le x < 360^\circ$ and give the general solution:

a) $\sqrt{3} + 2\cos x = 0$

b) $3\csc x - 6 = 0$

2.

Solve each of the following equations, $0 \le x < 2\pi$ and give the general solution:

a) $\cos^2 x - 3\cos x + 2 = 0$

b) $\tan^2 x = \tan x + 2$

3.

Solve over the real numbers. Give exact values wherever possible:

a) $\cos^2 A - 5\cos A + 1 = 0$

b) $\cot^2 x + 4\cot x - 5 = 0$

c) $6\sin^2 x + \sin x = 1$

d) $\tan x - 3\tan x \sin x = 0$

e) $2\sin^2 x - \sin x = 1$

f) $\sin 2x + \cos x = 0$

g) $2\sin x = 7 - 3\csc x$

4.

Solve $2\sin^2 x - \sin x = 1$ **exactly over each of the following domains:**

a) $\pi \le x < 2\pi$

b) $0 \le x < \dfrac{\pi}{2}$

c) $-\pi \le x < 0$

5.

Solve $\sin x - \cos 2x = 0$ **,** $0 \le x < 2\pi$ **and also give the general solution.**