CALCULUS 12 FINAL EXAM PREP.

KEY

Name ______________________ Date ______________

UNIT 1

***All** working must be shown, as applicable to obtain full marks*

1) Find the first derivative of each of the following functions:

a) $y = -6x^3 - 5$

$-18x^2$

b) $y = \pi x^4 + 8x^2 - 3x + 6$

$4\pi x^3 + 16x - 3$

c) $f(x) = 6\pi^2 x^3 - 4x^2 - 8\pi^3$

$18\pi^2 x^2 - 8x$

d) $g(x) = \frac{1}{3}x^6 - \frac{1}{2}x^8 + \frac{3}{5}x^5$

$2x^5 - 4x^7 + 3x^4$

2. If $y = 8p^2 - 2kp^3 + 5m^2$ find $\frac{dy}{dp}$

$16p - 6kp^2$

3. If $f(x) = 3x^3 - 2x^2 + 7$ find $f'(x)$ at the point (-1, 2)

$9x^2 - 4x$

$9(-1)^2 - 4(-1)$

$9 + 4 = 13$

4. If $y = 8x^4 - 6x + 3$ find : $y' = 32x^3 - 6$ or $\frac{dy}{dx} = 32x^3 - 6$

a) y''

$= 96x^2$

b) $\left(\frac{dy}{dx}\right)^2$ $(32x^3 - 6)^2$

c) $\frac{d^3y}{dx^3}$ $= 192x$

5) If $y = 4mx^3 - 6p^2x^2 - 8m + 3$ find:

a) $\frac{dy}{dx}$ b) $\frac{dy}{dm}$ c) $\frac{dy}{dp}$ d) $\frac{dy}{dw}$

$12mx^2 - 12p^2x$ $4x^3 - 8$ $-12px^2$ 0

6) Given $y = 3x^4 - 6x + 5$ find $\frac{dy}{dx}$ at x = -1

$12x^3 - 6$

$12(-1)^3 - 6$

$-12 - 6$ $= -18$

7) Find the first derivative of each of the following functions:

a) $y = \frac{6}{x^3} - \frac{4}{x^2} + \frac{16}{x} - 8$

RW $6x^{-3} - 4x^{-2} + 16x^{-1}$

$-18x^{-4} + 8x^{-3} - 16x^{-2}$

b) $y = 6\sqrt{x} + \frac{8}{\sqrt{x}} - \frac{15}{2\sqrt[3]{x}}$

RW $6x^{1/2} + 8x^{-1/2} - \frac{15}{2}x^{-1/3}$

$3x^{-1/2} - 4x^{-3/2} + \frac{5}{2}x^{-4/3}$

c) $f(x) = \frac{5x^2}{1-3x^3}$

$\frac{BT' - B'T}{B^2}$

$\frac{(1-3x^3)(10x) - (-9x^2)(5x^2)}{(1-3x^3)^2}$

d) $g(x) = \sqrt{6x^2 - 2x + 8}$

RW $(6x^2 - 2x + 8)^{1/2}$

$\frac{1}{2}(6x^2 - 2x + 8)^{-1/2}(12x - 2)$

8) Find the slope of the tangent line to $y = x^2 - 5x - 4$ at (-1, 2)

$2x - 5$

$2(-1) - 5$ $\quad m_T = -7$

9) Use implicit differentiation to find $\frac{dy}{dx}$ in terms of x and y:

a) $8x^2 - 4y^2 = 16$

$16x - 8yy' = 0$

$\frac{16x}{8y} = \frac{8yy'}{8y}$

$y' = \frac{dy}{dx} = \frac{8x}{4y} = \boxed{\frac{2x}{y}}$

b) $x^2 + 4xy - y^2 = 10$

$f\,s \qquad fs' + f's$

$2x + (4x)(y') + (4)(y) - 2yy' = 0$

$2x + 4xy' + 4y - 2yy' = 0$

$4xy' - 2yy' = -2x - 4y$

$\frac{y'(4x - 2y)}{4x - 2y} = \frac{-2x - 4y}{4x - 2y}$

$y' = \frac{dy}{dx} = \frac{-x - 2y}{2x - y}$

or $\frac{x + 2y}{y - 2x}$

10) Given the position function $s(t) = 6t^3 - 8^2 t + 9$ find the velocity and acceleration as functions of time t.

$v(t) = 18t^2 - 16t$

$a(t) = 36t - 16$

11) How fast is the side of a square growing when the length of the side is 5 m and the area is increasing at 0.75 m^2/s.

$A = s^2$

$\frac{dA}{dt} = 2s\frac{ds}{dt}$

$0.75 = 2(5)\frac{ds}{dt}$

$\frac{ds}{dt} = 0.075 m/s$

$s = 5$

$\frac{dA}{dt} = 0.75$

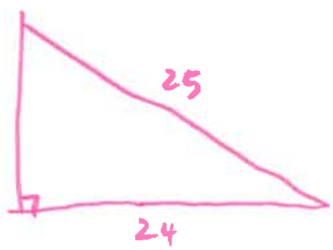
$\frac{ds}{dt} = ?$ @ 5m

12) The hypotenuse of a right triangle is of fixed length but the lengths of the other two sides x and y depend on time. How fast is y changing when $\frac{dx}{dt} = 8$ and $x = 24$ if the length of the hypotenuse is 25?

$x^2 + y^2 = z^2$

$y = \sqrt{25^2 - 24^2}$

$y = 7$



$\frac{dy}{dt} = ?$ $\quad \frac{dx}{dt} = 8$

$x = 24$

$y = 7$

$z = 25$ fixed

$x^2 + y^2 = z^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$2(24)(8) + 2(7)\frac{dy}{dt} = 0$

$\to 384 + 14\frac{dy}{dt} = 0$

$\frac{dy}{dt} = -27.43$

13) Differentiate each function:

a) $h(x) = 6\cos(4x)$

$-6\sin(4x)(4)$

or

$-24\sin(4x)$

b) $y = \sin(2x^4 + 3x)$

$\cos(2x^4 + 3x)(8x^3 + 3)$

or

$(8x^3 + 3)\cos(2x^4 + 3x)$

c) $y = \sin^3 4x + 4\tan 5x$

$(\sin 4x)^3 + 4\tan 5x$

$3(\sin(4x))^2(\cos(4x)(4)) + 4\sec^2(5x)5$

or

$12\sin^2(4x)\cos(4x) + 20\sec^2(5x)$

d) $f(x) = \overset{f}{4x}\overset{s}{\sin x}$ $\quad fs' + f's$

$(4x)(\cos x) + (4)\sin x$

14) Find the first derivative of each of the following functions:

a) $y = 3\ln\left(x^2 - 3\right)$

$\frac{3}{x^2-3} \cdot 2x$

or

$\frac{6x}{x^2-3}$

b) $k(x) = 4e^x - 3e^2 + 6e^\pi - 3e^{\pi x}$

$4e^x - 3e^{\pi x} \cdot \pi$

or

$4e^x - 3\pi e^{\pi x}$

c) $f(x) = 12^{x^2}$

$\ln(12) \cdot 12^{x^2}(2x)$

d) $y = 3e^x \ln x$

f s

$3e^x \cdot \frac{1}{x} + 3e^x \ln x$

or

$\frac{3e^x}{x} + 3e^x \ln x$

15) Differentiate each of the following functions:

a) $y = Bx^3 - 3Cx^3 + D^4$

$3Bx^2 - 9Cx^2$

b) $f(x) = 4\cos\frac{5}{x^2}$

RW $4\cos(5x^{-2})$

$-4\sin(5x^{-2})(-10x^{-3})$

c) $m(x) = \frac{x^2 - 5}{x^2 + 5}$

$\frac{BT' - B'T}{B^2}$

$\frac{(x^2+5)(2x) - (2x)(x^2-5)}{(x^2+5)^2}$

d) $y = y^2 + xy$

$y' = 2yy' + xy' + y$

$y' - 2yy' - xy' = y$

$\frac{y'(1 - 2y - x)}{1-2y-x} = \frac{y}{1-2y-x}$

$y' = \frac{y}{1-2y-x}$

UNIT 2

***All** working must be shown, as applicable to obtain full marks*

1) Find the general antiderivative of each of the following functions:

a) $5x^3 - 4x^2 + 3$

$\frac{5}{4}x^4 - \frac{4}{3}x^3 + 3x + C$

b) $x^4 + \frac{1}{x^3} + 2\pi$

$x^4 + x^{-3} + 2\pi$

$\frac{1}{5}x^5 - \frac{1}{2}x^{-2} + 2\pi x + c$

c) $\frac{12}{x} + 3\cos x - 2$

$12\ln x + 3\sin x - 2x + C$

d) $\cos(4x) + 4x^2$

$\frac{1}{4}\sin(4x) + \frac{4}{3}x^3 + c$

e) $6e^x - 4e^{3x}$

$6e^x - \frac{4}{3}e^{3x} + c$

f) $x^3\cos(x^4)$

$u = x^4$

$du = 4x^3\,dx$

$dx = \frac{du}{4x^3}$

$\int \cancel{x^3}\cos u \,\frac{du}{4\cancel{x^3}}$

$\frac{1}{4}\int \cos u\, du \Rightarrow \frac{1}{4}\sin(x^4) + c$

2) Find the position function $s(t)$ for an object with acceleration function $a(t) = 2t - 2$ and initial velocity $v(0) = 1$ and initial position $s(0) = 2$

$v(t) = t^2 - 2t + 1$

$s(t) = \frac{1}{3}t^3 - t^2 + t + 2$

3) Find each of the following indefinite integrals:

a) $\int x^3\, dx$

$\frac{1}{4}x^4 + c$

b) $\int \frac{2}{1-x}\, dx$

RW $2(1-x)^{-1}$

$2\ln(1-x)(-1)$

or

$-2\ln(1-x) + c$

c) $\int e^{-3x}\, dx$

$-\frac{1}{3}e^{-3x} + c$

d) $\int \cos 3x\, dx$

$\frac{1}{3}\sin 3x + c$

4) Find the **exact** value of each of the following definite integrals:

a) $\int_0^{\frac{\pi}{2}} \sin x \, dx$

$-\cos x \Big|_0^{\pi/2}$

$[-\cos \tfrac{\pi}{2}] - [-\cos 0]$

$0 \quad - \quad -1$

$= 1$

b) $\int_{-2}^{2} xe^{x^2} \, dx$

$u = x^2$
$du = 2x\,dx$
$dx = \frac{du}{2x}$

$\int_{-2}^{2} \cancel{x} e^{u} \frac{du}{2\cancel{x}}$

$\frac{1}{2}\int_{-2}^{2} e^{u} du \Rightarrow \frac{1}{2} e^{x^2} \Big|_{-2}^{2}$

$[\tfrac{1}{2} e^4] - (\tfrac{1}{2} e^4) = 0$

5) Simplify: [2 marks each]

a) $\int_0^a e^x \, dx$

$e^x \Big|_0^a \quad e^a - 1$

b) $\int_0^b 3x^3 \, dx$

$\frac{3}{4}x^4 \Big|_0^b = \frac{3}{4}b^4 - 0$

$= \frac{3}{4}b^4$

6) Simplify: $\int_1^4 3m^2 \, dz$ [2 marks]

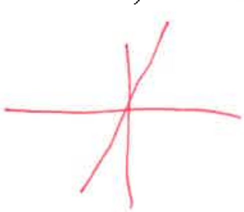
$3m^2 z \Big|_1^4$

$3m^2(4) - 3m^2(1)$

$12m^2 - 3m^2$

$9m^2$

7) Find the **exact** area between the curve $y = 4x$ and the x-axis over the interval $-3 \le x \le 4$



~~$\int_{-3}^{4} 4x\,dx \quad 2x^2 \Big|_{-3}^{4} \quad [2(4)^2] - [2(-3)^2]$~~

~~$32 - 18 = 14$~~

$\int_{-3}^{0} 4x\,dx + \int_{0}^{4} 4x\,dx$

$18 \qquad 32$

$= 50$

8) Find the **exact** area between the curve $y = 2x^2$ and the **y-axis** over the interval $0 \le x \le 3$

BOX METHOD

$3\times 18 - \int_0^3 2x^2\,dx$

$54 - \frac{2}{3}x^3 \Big|_0^3$

$54 - \left[\frac{2}{3}(3)^3\right] - [0]$

$54 - 18 = 36$

INTEGRATION with Respect to y

$x = \left(\frac{1}{2}y\right)^{1/2}$ $u = \frac{1}{2}y$

$du = \frac{1}{2}dy$

$dy = 2du$

$\int_0^{18} u^{1/2}\cdot 2du$

$2\int_0^{18} u^{1/2}du \Rightarrow 2\cdot\frac{2}{3}u^{3/2}$

$\frac{4}{3}\left(\frac{1}{2}y\right)^{3/2}\Big|_0^{18}$

$\left[\frac{4}{3}\left(\frac{1}{2}(18)\right)^{3/2}\right] - [0] = 36$

9) Find the **exact** area between $f(x) = 2x - 3$ and $g(x) = x^2 + 3$ over the interval $-2 \le x \le 5$

-3 B $\quad$ 3 T

$\int_{-2}^{5} x^2+3-(2x-3)\,dx \Rightarrow \int_{-2}^{5} x^2-2x+6\,dx$

$\frac{1}{3}x^3 - x^2 + 6x \Big|_{-2}^{5}$

$\left[\frac{1}{3}(5)^3-(5)^2+6(5)\right] - \left[\frac{1}{3}(-2)^3-(-2)^2+6(-2)\right]$

$\frac{140}{3} - -\frac{56}{3} = \frac{196}{3}$

10) Simplify: [2 marks]

a) $\int_a^b 4x^2\,dx$

$\frac{4}{3}x^3 \Big|_a^b$

$\frac{4}{3}(b)^3 - \frac{4}{3}(a)^3$

b) $\int_0^b \frac{1}{1+x}\,dx$

$\ln(1+x)\Big|_0^b$

$\ln(1+b) + \ln(1)$

$\ln(1+b) + 0$

$\ln(1+b)$

11) Integrate:

a) $\int x^3\sqrt{x^4-3}\,dx$

$u = x^4 - 3$

$du = 4x^3\,dx$

$dx = \frac{du}{4x^3}$

$\int \cancel{x^3}(u)^{1/2} \cdot \frac{du}{4\cancel{x^3}}$

$\frac{1}{4}\int u^{1/2}\,du \Rightarrow \frac{1}{4}\cdot\frac{2}{3}u^{3/2} \Rightarrow \frac{1}{6}(x^4-3)^{3/2} + C$

b) $\int \frac{\sin x}{(\cos x)^5}\,dx$

$u = \cos x$

$du = -\sin x\,dx$

$dx = \frac{du}{-\sin x}$

$\int \frac{\cancel{\sin x}}{u^5}\cdot\frac{du}{-\cancel{\sin x}} \Rightarrow -\int u^{-5}\,du$

$-\frac{1}{4}u^{-4}$

$+\frac{1}{4}(\cos x)^{-4} + C$

c) $\int xe^{6x}\,dx$

$u = x$ $\quad v = \frac{1}{6}e^{6x}$

$du = 1$ $\quad dv = e^{6x}$

$\frac{1}{6}xe^{6x} - \frac{1}{6}\int e^{6x}$

$\frac{1}{6}xe^{6x} - \frac{1}{36}e^{6x} + C$

d) $\int x\sec^2 x\,dx$

12) Determine the area between $y = 3 - x$ and $x = 3y - y^2$.

$x = 3 - y$

L $\quad$ R

$3 - y = 3y - y^2$

$y^2 - 4y + 3 = 0$

$(y-3)(y-1)$

$3 \quad 1$

$\int_1^3 3y - y^2 - (3-y)\,dy$

$\int_1^3 -y^2 + 4y - 3\,dy$

$-\frac{1}{3}y^3 + 2y^2 - 3y \Big|_1^3$

$\left[-\frac{1}{3}(3)^3 + 2(3)^2 - 3(3)\right] - \left[-\frac{1}{3}(1)^3 + 2(1)^2 - 3(1)\right]$

$0 - -\frac{4}{3} = \frac{4}{3}$

UNIT 3

***All** working must be shown, as applicable to obtain full marks*

1) Evaluate each limit:

a) $\lim_{x \to -2} 3x^2 + 1$

$3(-2)^2 + 1$

13

b) $\lim_{x \to 3} \frac{2x^2+1}{3x}$

$\frac{2(3)^2+1}{3(3)} = \frac{19}{9}$

c) $\lim_{x \to 3\pi} \left(x^2 + 6\pi x - 2\pi^2\right)$

$(3\pi)^2 + 6\pi(3\pi) - 2\pi^2$

$9\pi^2 + 18\pi^2 - 2\pi^2$

$25\pi^2$

d) $\lim_{x \to a} \frac{(x+2a)^2}{x^2+a^2}$

$\frac{(a+2a)^2}{a^2+a^2} = \frac{(3a)^2}{2a^2} = \frac{9\cancel{a^2}}{2\cancel{a^2}}$

$\frac{9}{2}$

2. By evaluating one-sided limits, find the indicated limit if it exists:

$$f(x) = \begin{cases} x^2 = 4 & \text{if } x < 2 = \\ -2x+5 = 1 & \text{if } x \geq 2 = \end{cases} \quad \text{find } \lim_{x \to 2} f(x) = \text{DNE}$$

3. Evaluate each limit.

a. $\lim_{x \to -3} f(x) = 0$

b. $f(1) = \text{DNE}$

c. $\lim_{x \to 1} f(x) = -5$

d. $\lim_{x \to -2^+} f(x) = 4$

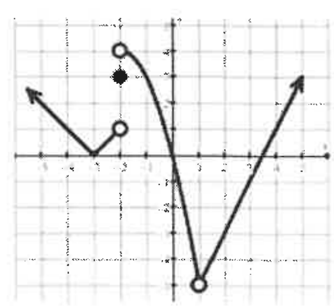
e. $f(3) = -1$

f. $\lim_{x \to -2^-} f(x) = 1$

g. $\lim_{x \to -2} f(x) = \text{DNE}$

h. $f(-2) = 3$

i. $f(4) = 1$



4. Evaluate each of the following limits using L'Hopital's Rule:

a) $\lim_{x\to 3} \frac{9-x^2}{x-3}$ $= \frac{0}{0}$

L'H $\frac{-2x}{1} = -6$

b) $\lim_{x\to 0} \frac{4\sin 2x}{4x}$ $\frac{0}{0}$

L'H $\frac{4\cos(2x)(2)}{4}$

$\frac{8\cos 0}{4} = \frac{8}{4} = 2$

c) $\lim_{x\to 1} \frac{\ln x}{x-1}$ $\frac{0}{0}$ L'H

$\frac{\frac{1}{x}\cdot x}{1\cdot x} = \frac{1}{x} = \frac{1}{1}$

1

d) $\lim_{x\to 0} \frac{1-e^x}{\sin x}$ $\frac{0}{0}$ L'H $\frac{-e^x}{\cos x} = \frac{-1}{1}$

-1

5) Evaluate the following limit by factoring and simplifying: $\lim_{x\to 5} \frac{x^2-25}{x-5}$

$\frac{(x+5)\cancel{(x-5)}}{\cancel{(x-5)}}$

$\lim_{x\to 5} 5+5 = 10$

6) Evaluate the following limit by rationalizing the numerator: $\lim_{x\to 3} \frac{\sqrt{3}-\sqrt{x}}{3-x} \cdot \frac{(\sqrt{3}+\sqrt{x})}{(\sqrt{3}+\sqrt{x})}$

$$\frac{(\cancel{3-x})}{(\cancel{3-x})(\sqrt{3}+\sqrt{x})}$$

$$\lim_{x\to 3} \frac{1}{\sqrt{3}+\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

7) Evaluate each of the following limits:

a) $\lim_{x\to 2\pi} \left(2x^2 - 6\pi x\right)$

$2(2\pi)^2 - 6\pi(2\pi)$

$8\pi^2 - 12\pi^2$

$-4\pi^2$

b) $\lim_{x\to 4} \frac{16-x^2}{4-x}$ $\quad \frac{0}{0}$ L'H $\frac{-2x}{-1}$ $\frac{-8}{-1}$

8

c) $\lim_{x\to 0} \frac{2x}{e^x - 1}$ $\quad \frac{0}{0}$

L'H $\quad \frac{2}{e^x} = \frac{2}{1}$

2

d) $\lim_{t\to -2} \frac{t^2 - 4t}{t-2}$ $\quad \frac{4-4(-2)}{-2-2} = \frac{4+8}{-4}$ $\frac{12}{-4}$

-3

e) $\lim_{x\to a} \frac{5x+a}{5x-a}$

$\frac{5a+a}{5a-a} = \frac{6a}{4a}$

$\frac{3}{2}$

f) $\lim_{x\to 4} \frac{\ln(5-x)}{x-4}$ $\quad \frac{0}{0}$

L'H $\quad \frac{\frac{-1}{(5-x)}}{1}$ $\quad \frac{-1}{5-4} = \frac{-1}{1}$

-1

8) Use the **definition** of the derivative to find $\frac{dy}{dx}$ for $2x^2 + x$

$$\lim_{h\to 0} \frac{2(x+h)^2 + (x+h) - (2x^2+x)}{h}$$

$$\frac{2(x^2+2xh+h^2) + x+h - 2x - x}{h}$$

$$\lim_{h\to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 \cancel{+x} + h - \cancel{2x^2} - \cancel{x}}{h}$$

$$\lim_{h\to 0} \frac{4xh + 2h^2 + h}{h} \Rightarrow \frac{\cancel{h}(4x+2h+1)}{\cancel{h}}$$

$$\lim_{h\to 0} 4x + 2h + 1 = 4x+1$$

9) Use the **definition** of the derivative to find $\frac{dy}{dx}$ for $y = \frac{1}{x+4}$

$$\lim_{h\to 0} \frac{\frac{1}{x+h+4} + \frac{-1}{x+4}}{h}$$

$$\frac{\cancel{x}\cancel{+4} - \cancel{x} - h - \cancel{4}}{h(x+h+4)(x+4)}$$

$$\lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(x+h+4)(x+4)}$$

$$\lim_{h\to 0} \frac{-1}{(x+h+4)(x+4)} = \frac{-1}{(x+4)^2}$$

UNIT 4

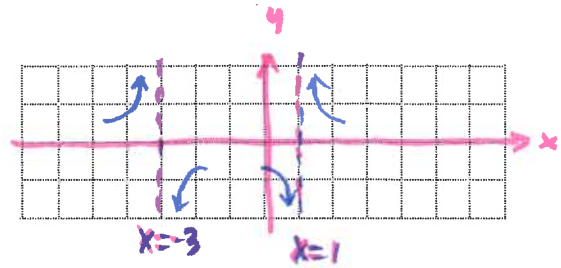
***All* working must be shown, as applicable to obtain full marks**

1. Find the vertical asymptotes of the following function and sketch the graph near the asymptotes: $f(x) = \dfrac{4}{(x-1)(x+3)}$

VA $x = 1$, $x = -3$

-3 LS	RH
-3.001	-2.999

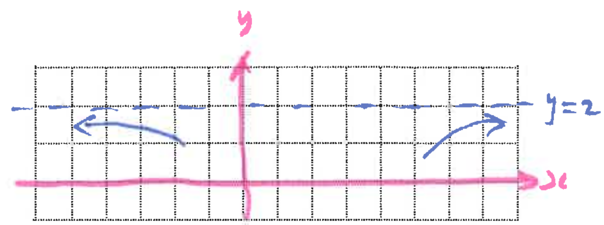
1 LS	RS
0.999	1.0001



2. Find the horizontal asymptotes of the following function and sketch the graph near the asymptotes: $y = \dfrac{2x^2-5}{x^2+7}$

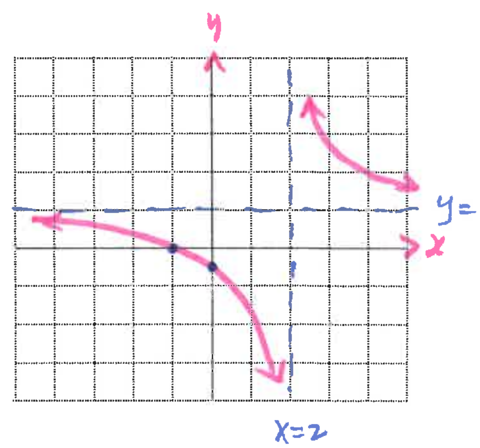
HA = 2

$-\infty$	∞
-99	99



3. Make a rough sketch of the following function. Indicate x-intercepts, y-intercepts, vertical and horizontal asymptotes: $y = \dfrac{x+1}{x-2}$

xint	yint
$x+1=0$, $x=-1$	$\dfrac{0+1}{0-2} = -\dfrac{1}{2}$
VA $x-2=0$, $x=2$	HA $y=1$



4. Use Calculus and the Second Derivative Test to determine Concave up, Concave down, and Inflection Point for the following. **Sketch** function to show the above features.

$$f(x)=\frac{1}{x^2+1}$$

$f'(x) = (x^2+1)^{-1}$

$-1(x^2+1)^{-2}(2x)$

$f'(x) = \dfrac{-2x}{(x^2+1)^2} = 0$

C.P: $x = 0$

$f''(x) = (-2x)(x^2+1)^{-2}$ $fs'+fs'$

$(-2x)(-2(x^2+1)^{-3}(2x)) + (-2)(x^2+1)^{-2}$

$\dfrac{8x^2}{(x^2+1)^3} + \dfrac{-2}{(x^2+1)^2} \cdot \dfrac{(x^2+1)}{(x^2+1)}$

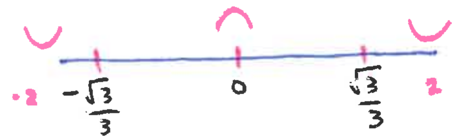
$f''(x) = \dfrac{8x^2-2x^2-2}{(x^2+1)^3} = 0$

I.P $6x^2-2=0$

$\dfrac{6x^2}{6} = \dfrac{2}{6}$

$\sqrt{x^2} = \sqrt{\dfrac{1}{3}}$

I.P: $x = \pm\dfrac{\sqrt{3}}{3}$



$f''(0) = \frac{-}{+}$ $-$ ∩

$f''(-\frac{\sqrt{3}}{3}) = \frac{+}{+}$ $+$ ∪

$f''(\frac{\sqrt{3}}{3}) = \frac{+}{+}$ $+$ ∪

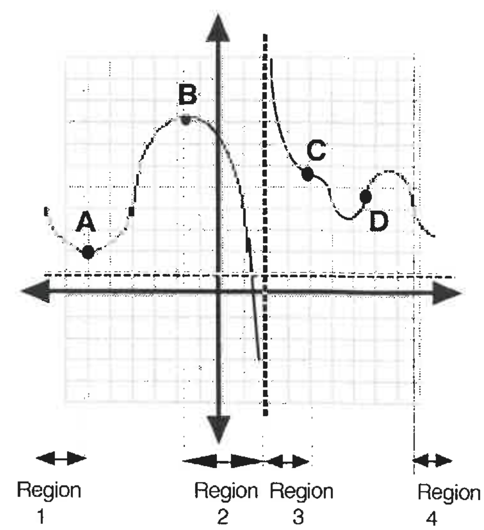
5. Use the terms increasing, decreasing, concave up, concave down to describe each region below:

Region 1 Decr CU

Region 2 Decr CD

Region 3 Decr CU

Region 4 Decr CU



b) Use the terms local max/min, global man/min, turning point, inflection point to describe each of the following points:

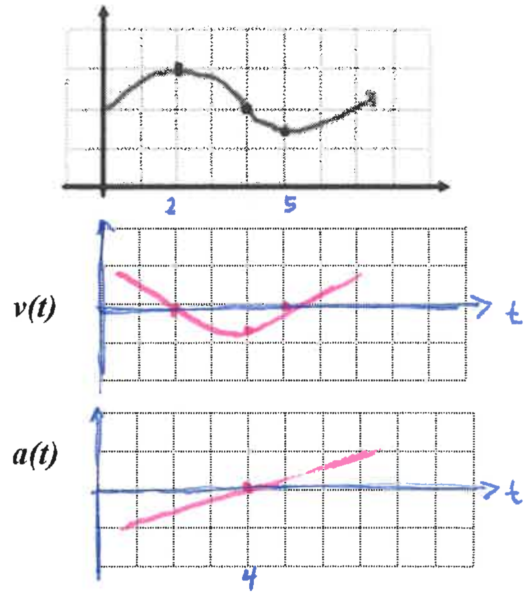
A L.Min B L.Max C I.P. D I.P.

c) give the equations of **all** asymptotes:

VA: $x = 2$

HA: $y = 1$

6. Given the position-time graph below:



a) at what time(s) is the velocity 0?

2 & 5 sec.

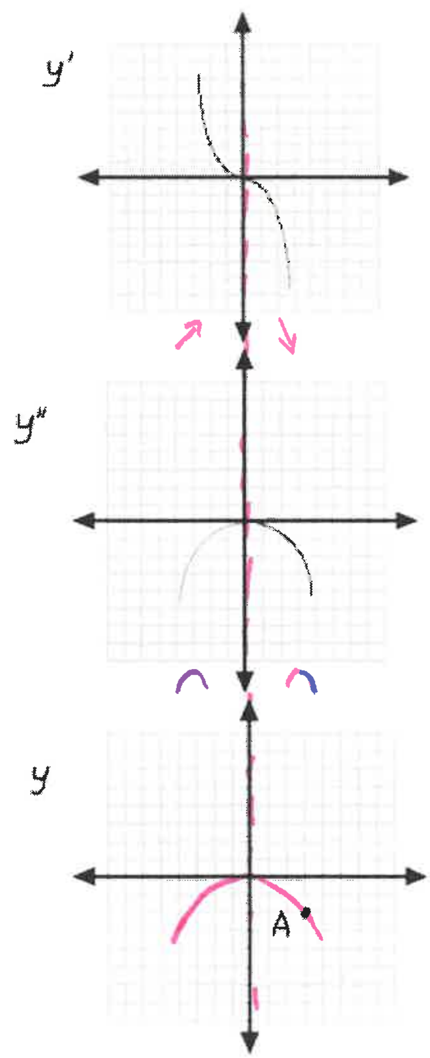
b) at what time(s) is the acceleration 0?

4 sec.

c) when is the object slowing down?

(0, 2 sec) & (4, 5 sec)

7. Given the graphs of the first derivative y'
and the second derivative y",
sketch the graph of y passing through A.



UNIT 5

***All* working must be shown, as applicable to obtain full marks**

1. a) $4 \ln x - 12 = 0$

$\frac{4\ln x}{4} = \frac{12}{4}$

$e^{\ln x} = e^3$ $\quad x = e^3$

b) $2 \ln x = \ln(2x + 8)$

$\cancel{\ln} x^2 = \cancel{\ln}(2x+8)$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2)$

$x = 4$ $\quad x = -2$ Rej

2. Simplify:

a) $\ln(e^{-4}) + \ln(e^{6})$

$\ln(e^{-4} \cdot e^{6})$

$\ln e^2$

$2 \ln(e)$

b) $e^{-2\ln 4}$

$e^{\ln 4^{-2}}$

$4^{-2} = \frac{1}{16}$

3. Find the equation of the tangent line to the curve y = 2 ln x at the point (e, 2).

$y' = \frac{2}{x}$ $\quad m_T = \frac{2}{e}$

$y = mx + b$

$2 = \frac{2}{\cancel{e}} \cdot \cancel{e} + b$

$b = 0$ $\quad \therefore \quad y = \frac{2}{e}x$

4. Use Newton's Method to solve the following equation $x^3 - 2x^2 - 2 = 0$ using x-initial value $x = 2$. Just do two iterations $x_3 =$

$f(x) = x^3 - 2x^2 - 2$

$f'(x) = 3x^2 - 4x$

$x_1 = 2$

$x_2 = 2 - \frac{(2)^3 - 2(2)^2 - 2}{3(2)^2 - 4(2)} = \frac{5}{2}$

$x_3 = \frac{5}{2} - \frac{(\frac{5}{2})^3 - 2(\frac{5}{2})^2 - 2}{3(\frac{5}{2})^2 - 4(\frac{5}{2})}$

$x_3 = 2.3714$

5. An open field is bounded by a lake with a straight shoreline. A rectangular enclosure is to be constructed using 600 m of fencing along three sides and the lake as a natural boundary on the fourth side. What dimensions will maximize the enclosed area and what is the maximum area? Solve this problem using Calculus.

Dimension

300m x 150m

Max Area

45,000m²

6. An open-top box is to be made from a 24 in. by 36 in. piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What size square should be cut out of each corner to get a box with the maximum volume?

The maximum volume is V(10−2√7)=640+448√7≈1825in.3 as shown in the following

cut off

$x = 10 - 2\sqrt{7}$ or 4.71

Max Vol.: 1825.3 in^3

7. Evaluate each of the following **IMPROPER INTEGRALS** if possible

a) $\int_1^{\infty} \frac{1}{x^2}\, dx$

$\int_1^{\infty} x^{-2}\, dx$

$-\frac{1}{x}\Big|_1^{\infty}$

$\left[\frac{-1}{\infty}\right] - \left[\frac{-1}{1}\right]$

$0 + 1 = 1$

b) $\int_{-\infty}^{0} 2e^x\, dx$

$2e^x\Big|_{-\infty}^{0}$

$\frac{2}{e^{\infty}} = 0$

$\left[2e^0\right] - \left[2e^{-\infty}\right]$

$2 - 0 = 2$

8. a) Use the MVT to determine all the numbers c for the following function:

$f(x) = x^3 + 4x$, $[-1, 1]$

$f'(x) = 3x^2 + 4$ a b: -5 5 $f(1) = 5$, $f(-1) = -5$ $\Rightarrow 3x^2 + 4 = \frac{5+5}{1+1} = \frac{\pm\sqrt{3}}{3}$

$\therefore C = \frac{\pm\sqrt{3}}{3}$

b) Use Rolle's Theorem to show that the function has a horizontal tangent line in the interval [0, 2] $f(x) = x^2 - 2x$

1. continuous ✓
2. different. ✓

$f(0) = 0$ ✓
$f(2) = 0$

$\therefore$ Rolle Theorem applies $2x - 2 = 0$, $x = 1$ $\therefore C = 1$

c) Use the IVT to show that $f(x) = x^4 - 7x^2 + 10$ has a root somewhere in the interval [0, 2]

$f(0) = 10$
$f(2) = -2$

Root lies between $\therefore$ IVT $x^4 - 7x^2 + 10 = 0$ Root $C = \sqrt{2}$

9. Given f is a function of x and y find the partial derivative of f with respect to x and the partial derivative of f with respect to y of each of the following:

$a)\ f(x,y) = 3x^2 + x^3y$

$\frac{\partial f}{\partial x} = 6x + 3x^2y$

$\frac{\partial f}{\partial y} = x^3$

$b)\ f(x,y) = x^2 lny - y^2$

$\frac{\partial f}{\partial x} = 2x\ln y$

$\frac{\partial f}{\partial y} = \frac{x^2}{y} - 2y$

10. Find the exact value of each of the following multiple integrals:

a) $\int_0^1 \int_2^4 x^2\,dx\,dy$

$\int_2^4 x^2\,dx \Rightarrow \frac{1}{3}x^3 \Big|_2^4$ $\left[\frac{1}{3}(4)^3\right] - \left[\frac{1}{3}(2)^3\right]$

$\frac{64}{3} - \frac{8}{3} = \frac{56}{3}$

$\int_0^1 \frac{56}{3}dy \Rightarrow \frac{56}{3}y$ $\left[\frac{56}{3}(1)\right] - [0] = \frac{56}{3}$

b) $\int_b^a \int_{-1}^5 \frac{x}{3}\,dx\,dy$ $\int_{-1}^5 \frac{1}{3}x\,dx \Rightarrow \frac{1}{6}x^2 \Big|_{-1}^5$

$\left[\frac{1}{6}(5)^2\right] - \left[\frac{1}{6}(-1)^2\right]$

$\frac{25}{6} - \frac{1}{6} = 4$

$\int_b^a 4\,dy$ $4y \Big|_b^a$

$4a - 4b$

11. Simplify:

a) $(3 - 4i) + (1 + 9i)$

$4 + 5i$

b) $(4 - 3i)(2 + 4i)$

$8 + 16i - 6i - 12i^2$

$8 + 10i + 12$

$= 10i + 20$

12. Solve each of the following equations over the COMPLEX NUMBERS:

$a)\ x^2 + 25 = 0$

$\sqrt{x^2} = \sqrt{-25}$

$x = \sqrt{25} \cdot \sqrt{-1}$

$x = \pm 5i$

$b)\ 2x^2 + 3 = x^2 - 1$

$\sqrt{x^2} = \sqrt{-4}$

$x = \sqrt{4} \cdot \sqrt{-1}$

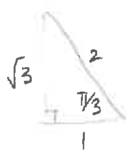
$x = \pm 2i$

13. Give the rectangular coordinates of the point whose polar coordinates are given:

$a)\ (r, \theta) = \left(3, \frac{\pi}{3}\right)$

$x = r\cos\theta \qquad y = r\sin\theta$

$r \quad \theta$



$x = 3\cos(\pi/3)$

$3\left(\frac{1}{2}\right)$

$x = 3/2$

$y = 3\sin \pi/3$

$3\left(\frac{\sqrt{3}}{2}\right)$

$y = \frac{3\sqrt{3}}{2}$

$\left(3/2, \frac{3\sqrt{3}}{2}\right)$

$b)\ (r, \theta) = (6, \pi)$

$x = 6\cos(\pi)$

$6(-1)$

$x = -6$

$y = 6\sin\pi$

$6(0)$

$y = 0$

$(-6, 0)$

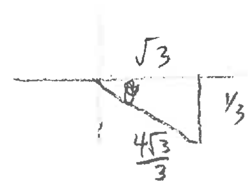
14. Give the polar coordinates of the point whose rectangular coordinates are given:

DONE IN RADIANS

$a)\ (x, y) = (5,5)$

$(r, \theta) = 5\sqrt{2}, \frac{\pi}{4}$

$b)\ (x, y) = \left(\sqrt{3}, -\frac{1}{3}\right)$



$r = \sqrt{(\sqrt{3})^2 + \left(\frac{1}{3}\right)^2}$

$r = \sqrt{\frac{10}{3}}$

$r = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$

arctan $\tan\theta =$ arctan $\frac{1/3}{\sqrt{3}}$

$\tan^{-1}(1/3 / \sqrt{3}) = 0.524_r$

$2\pi -$

(r, θ)

$\left(\frac{2\sqrt{7}}{3}, 6.09\right)$

15. Find a rectangular equation equivalent to the given polar equation and describe the graph:

a) $r^2 = 2^2$

$r^2 = 4$

$x^2 + y^2 = 4$

circle with centre (0,0) radius 2

b) $r = 4\cos\theta$

$r \cdot r = \frac{4x}{r} \cdot r$

$r^2 = 4x$

$x^2 + y^2 = 4x$

$(x^2 - 4x) + y^2 = 0$

$(x-2)^2 + y^2 = 4$

circle with Centre (2,0) $r = 2$