CALCULUS 12

PRACTICE QUESTIONS

for

FINAL

Solutions

Frances Kelsey School

Part I — Multiple-Choice Problems

Instructions: Write the letter corresponding to each of your answers in the blank box that is provided. Correct answers do not require work to receive full credit. However, partial credit can be awarded for incorrect answers based on the work that is shown in the adjacent blank spaces. Hence, you are strongly advised to show your work for each problem.

$y=mx+b$

(1) **[10 points]** Determine which of the following is an equation of the tangent line to the curve $y = \sqrt{x}$ at the point $(9, 3)$.

(A) $y = 6x - 51$.

(B) $y = 3x + 24$.

(C) $y = \frac{1}{6}x + \frac{3}{2}$.

(D) $y = \frac{\sqrt{x}}{2} - \frac{9}{2\sqrt{x}} + 3$.

Answer: C

$y' = \frac{1}{2}x^{-1/2}$

$m_T = \frac{1}{2\sqrt{x}} \Rightarrow \frac{1}{2\sqrt{9}} = \frac{1}{6}$

$3 = \frac{1}{6}(9) + b$

$b = 3/2 \quad \therefore \quad y = \frac{1}{6}x + \frac{3}{2}$

(2) **[10 points]** If $\overset{F\ S}{x^2y} + \overset{F\ S}{xy^2} = 3x$, then $\frac{dy}{dx}$ is

(A) $\frac{x2 + xy^2}{3}$.

(B) $\frac{3 - 2xy - y^2}{x^2 + 2xy}$.

(C) $2x^2y + y^2$.

(D) $\frac{2x+3}{x^2+x}$.

Answer: B

$x^2y' + 2xy + x2yy' + y^2 = 3$

$x^2y' + 2xy' = 3 - 2xy - y^2$

$\frac{y'(x^2+2xy)}{x^2+2xy} = \frac{3-2xy-y^2}{x^2+2xy}$

(3) [**10 points**] $\int_0^\pi sin(t)dt$

(A) -2

(B) 0

(C) 2

(D) $\frac{3\pi}{4}$

Answer: C

$-\cos t \Big|_0^{\pi}$

$-100 \quad \pi \qquad$ CST 100

CST

$-\cos(\pi) - -\cos(0)$

$-(-1) + (1)$

$1 + 1 = 2$

(4) [**10 points**] Evaluate $\lim_{x\to 0} \frac{1-\cos(4x)}{x^2}$.

(A) 8.

(B) 4.

(C) 2.

(D) 1.

Answer: A

L'H $\qquad\qquad$ L'H

$\frac{d/dx \; -\cos(4x)}{d/dx \; x^2} \Rightarrow \frac{4\sin 4x}{2x} = \frac{16\cos 4x}{2}$

$\lim_{x\to 0} \frac{16\cos 4(0)}{2} = \frac{16}{2} = 8$

(5) [**10 points**] Evaluate $\int_{-2}^{1} x^2 - 1 \, dx$.

(A) $-\frac{4}{3}$

(B) 0

(C) $\frac{3}{2}$.

(D) $\frac{4}{3}$

$$\frac{1}{3}x^3 - x \Big|_{-2}^{1}$$

$$\left[\frac{1}{3}(1)^3 - (1)\right] - \left[\frac{1}{3}(-2)^3 - (-2)\right]$$

$$= 0$$

Answer: $\boxed{B}$

(6) [**10 points**] Find the largest open interval on which $f(x) = \overset{F}{x}\overset{S}{e^x}$ is concave upward.

(A) $(0, \infty)$.

(B) $(-1, \infty)$.

(C) $(-2, \infty)$.

(D) $(-\infty, \infty)$.

Answer: $\boxed{C}$

C.P $f'(x) = xe^x + e^x$

$f'(x) = e^x(x+1)$

$0 = e^x(x+1)$

$\cancel{e^x} = 0 \quad x + 1 = 0$

$x = -1$

C.P: $(-1, -0.4)$ ∨

I.P. $f''(x) = xe^x + e^x + e^x$

$f''(x) = xe^x + 2e^x$

$f''(x) = e^x(x+2)$

$0 = e^x(x+2)$

I.P: $(-2, -0.3)$

Concavity Test

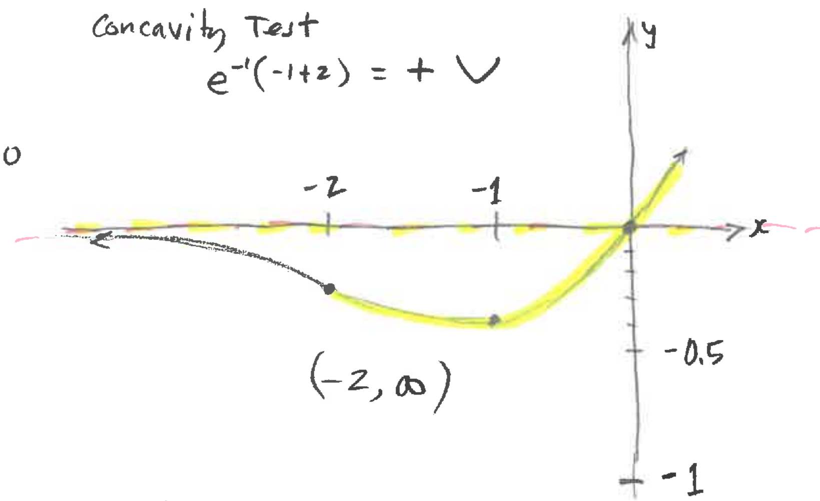
$e^{-1}(-1+2) = +$ ∨

y int ⇒ Let $x = 0$

$y = 0e^0$

y int $= 0$ ∴ x int $= 0$

Also e^x exponential function ∴ HA $y = 0$



(7) [**10 points**] Determine which of the following equals $\int x\sqrt{x^2+1}\,dx$.

(A) $\frac{1}{3}x^2(x^2+1)^{3/2}+c$.

(B) $\frac{1}{3}(x^2+1)^{3/2}+c$.

(C) $\frac{1}{2}x^2(x^2+1)^{3/2}+c$.

(D) $\frac{1}{2}(x^2+1)^{3/2}+c$.

Answer: $\boxed{B}$

$u = x^2+1$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$\int \cancel{x}\, u^{1/2} \frac{du}{2\cancel{x}}$

$\frac{1}{2}\int u^{1/2}\,du$

$\downarrow$

$\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} u^{3/2}$

$\frac{1}{3}(x^2+1)^{3/2} + c$

(8) [**10 points**] Let f and g be differentiable functions defined on $(-\infty, \infty)$. Suppose we have the following table of values for f, g, f' and g'.

x	0	1	2	3	4	5
$f(x)$	-7	30	4	-10	8	6
$g(x)$	14	7	-11	2	36	12
$f'(x)$	1	-4	5	0	30	8
$g'(x)$	18	-3	24	4	6	-2

Using this table, find the value of $(f \circ g)'(3)$.

Chain Rule

(A) -20.

(B) 0.

(C) 20.

(D) 32.

Answer: $\boxed{C}$

$f'(g(x)) \cdot g'(x)$

$f'(g(3)) \cdot g'(3)$

$f'(2) \cdot g'(3)$

$5 \cdot 4 = 20$

I.B.P ILATE

(9) [**10 points**] Determine which of the following equals $\int x^2 \ln(x)\, dx$.

(A) $\frac{1}{3}x^3 \ln(x) - \frac{1}{9}x^3 + c.$

(B) $\frac{1}{3}x^3 \ln(x) - \frac{1}{6}x^3 + c.$

(C) $\frac{1}{3}x^3 \ln(x) - \frac{1}{9}x^2 + c.$

(D) $\frac{1}{3}x^3 \ln(x) - \frac{1}{6}x^2 + c.$

Answer: $\boxed{A}$

$u = \ln x \qquad V = \frac{1}{3}x^3$

$du = \frac{1}{x} \qquad dv = x^2$

$\frac{1}{3}x^3 \ln x - \int \frac{1}{3} x^2 dx$

$'' \quad - \frac{1}{3}\int x^2 dx$

$'' \quad - \frac{1}{3} \cdot \frac{1}{3}x^3$

$\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$

(10) [**10 points**] Determine the value of the definite integral $\int_0^4 \frac{1}{\sqrt{x}} dx$

(A) ∞.

(B) 4.

(C) 5.

(D) 3.

Answer: $\boxed{B}$

$\int_0^4 x^{-1/2}$

$2x^{1/2} \Big|_0^4$

$2\sqrt{4} - 2\sqrt{0}$

$4 - 0 = 4$

(11) [**10 points**] If the volume V of a cube is decreasing at the rate of 24 in^3/sec, then find the rate at which the length of a side of the cube is decreasing when $V = 8$ in^3.

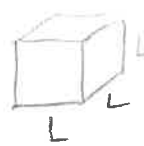
(A) 1 in/sec.

(B) 2 in/sec.

(C) 3 in/sec.

(D) 4 in/sec.

Answer: $\boxed{B}$



$V = L^3 \qquad \sqrt[3]{8} = \sqrt[3]{L^3} \quad L = 2$

$\frac{dv}{dt} = 3L^2 \frac{dL}{dt}$

$\frac{dv}{dt} = 24$

$24 = 3(2)^2 \frac{dL}{dt}$

$\frac{dL}{dt} = ?$ at $V = 8$

$L = 2$

$\frac{24}{12} = 2$ in/sec

(12) **[10 points]** Let $s(t)$ be the function of a mouse moving along the x-axis. Let $v(t)$ and $a(t)$ be its velocity and acceleration functions respectively. If

$$a(t) = 2 + 4e^{2t}, \quad v(0) = 1 \quad \text{and} \quad s(0) = 4,$$

determine which of the following expressions describes $s(t)$.

(A) $8e^{2t}$.

(B) $t^2 + e^{2t}$.

(C) $t^2 + 8e^{2t} - 3t - 4$.

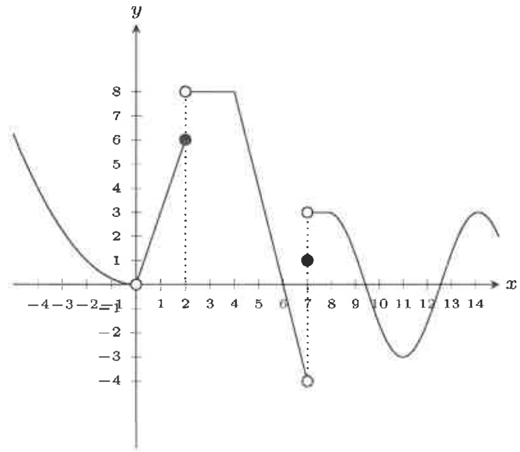
(D) $t^2 + e^{2t} + t + 4$.

$v(t) = 2t + 2e^{2t} + 1$

$s(t) = t^2 + e^{2t} + t + 4$

Answer: $\boxed{D}$

(13) **[10 points]** The figure below shows part of the graph of a function f.



Using this figure, determine which of the following statements about f is <u>false</u>.

(A) $\lim_{x\to 0} f(x)$ exists.

(B) f is discontinuous at 2.

(C) It is continuous at 4.

(D) $\lim_{x\to 7} f(x)$ exists. $\Rightarrow$ false because $\left.\begin{array}{l}\lim_{x\to 7^-} f(x) = -4 \\ \lim_{x\to 7^+} f(x) = 3\end{array}\right\}$ not equal

Answer: $\boxed{D}$

(14) [**10 points**] Gven s(t) = $2t^3 - 3t^2 + 1$. Find the total distance traveled after 4 seconds.

$v(t) = 6t^2 - 6t$

$\rightarrow$ Now Find xint to break up integration

$6t(t-1)$

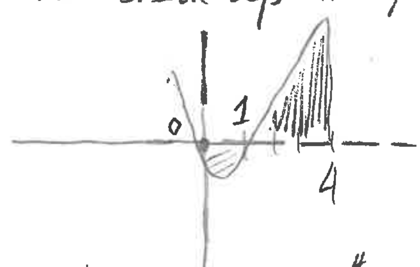
0 1

(A) 67

(B) 68

(C) 32

(D) 16



$\left| \int_0^1 v(t) \right| + \int_1^4 v(t)$

$= 68$

Answer: $\boxed{B}$

$y'' = 0$

(15) [**10 points**] An inflection point of the function $f(x) = 2x^3 - 9x^2 - 24x - 10$ is

(A) 1.

(B) 1.5.

(C) 3.

(D) 4.

Answer: $\boxed{B}$

$f'(x) = 6x^2 - 18x - 24$

$f''(x) = 12x - 18 = 0$

$\frac{12x}{12} = \frac{18}{12}$

$\frac{3}{2}$ or 1.5

(16) [**10 points**] Find the indefinite integral $\int \frac{3cos(ln(x))}{x} dx$ for $x > 0$.

(A) $3sin(ln(x)) + C$.

(B) $3cos(ln(x)) + C$.

(C) $3sec(ln(x)) + C$

(D) $3tan(ln(x)) + C$.

Answer: $\boxed{A}$

$u = \ln x$

$du = \frac{1}{x} dx$

$dx = x\,du$

$\int \frac{3\cos u}{\cancel{x}} \cdot \cancel{x} du$

$3 \int \cos u \, du$

$3 \sin u$

$3 \sin(\ln x) + c$

Part II — Show-Your-Work Problems

Instructions: Show all necessary work, and provide full justification for each answer. Circle your final answer(s).

(1) [**30 points**] If $f(x) = \dfrac{x^2 - 4x + 3}{x^2}$ then $f'(x) = \dfrac{4x - 6}{x^3}$ and $f''(x) = \dfrac{-8x + 18}{x^4}$.

$f'(x)$

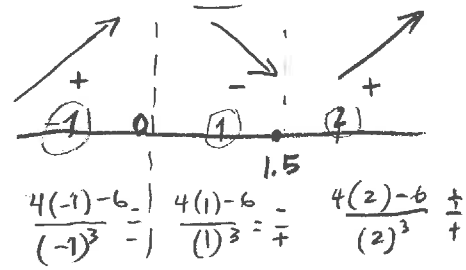
(a) Find the open intervals where f is increasing and where f is decreasing.

INC	DEC
$(-\infty, 0)$	$(0, 1.5)$
$(1.5, \infty)$	

VA: $x = 0$

$f'(x) = 4x - 6 = 0$

$x = \frac{3}{2}$ or 1.5



(b) Find the open intervals where f is concave upward and where f is concave downward.

$f''(x)$

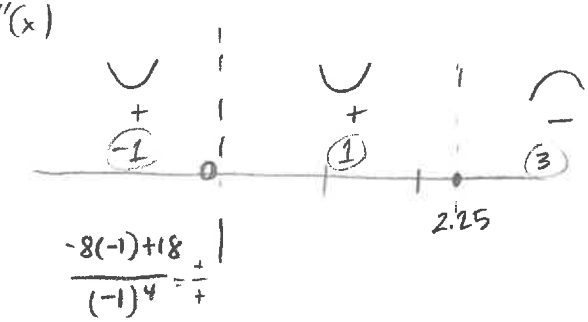
1^{st} Find I.P

$-8x + 18 = 0$

$x = \frac{9}{4}$ or

2.25

CU	CD
$(-\infty, 0)$	$(\frac{9}{4}, \infty)$
$(0, 9/4)$	

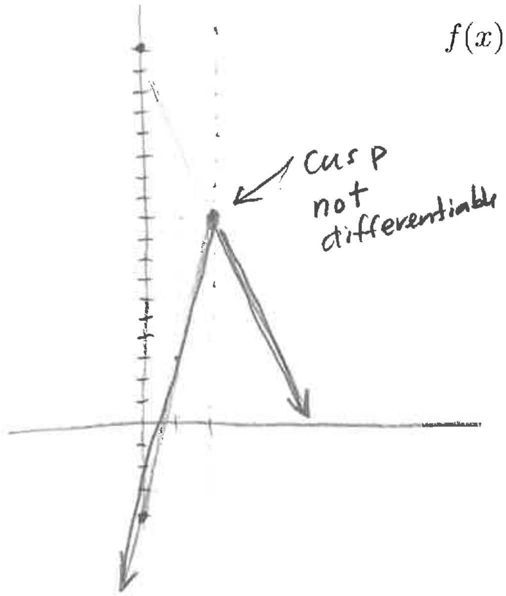


(c) Find all local minima and local maxima for f if they exist.

$\left(\frac{3}{2}, -\frac{1}{3}\right)$ See a

(2) [**30 points**] Find the value of the constant k for which the following piecewise-defined function is continuous everywhere. For the resulting function determine where the function is not differentiable. Justify your answers.

$$f(x) = \begin{cases} 7x + k & \text{if } x \leq 2; \\ 18 + kx & \text{if } x > 2. \end{cases}$$



$7(2) + K = 18 + 2K$

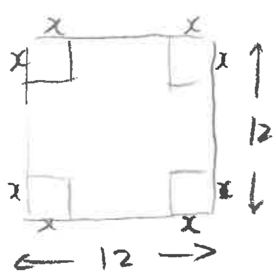
$14 + K = 18 + 2K$

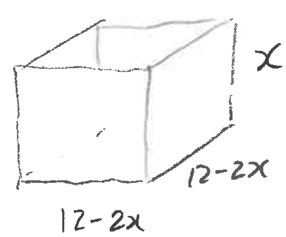
$-4 = K$

$\therefore\ 7x - 4$

$18 + -4x$

(3) [**30 points**] An open top box is to be made by cutting small identical squares from the corners of a 12×12 inch sheet of tin and bending up the remaining sides. How large should the squares cut from the corners be to make the box hold as much as possible (maximum volume)?





$V = x(12-2x)(12-2x)$

$V = 4x^3 - 48x^2 + 144x$

$V' = 12x^2 - 96x + 144$

$12(x^2 - 8x + 12) = 0$

$(x-6)(x-2)$

$\cancel{x=6}$ $\quad x = 2$

$\boxed{\text{Cutout } x = 2}$

B   T

-3 $+1$

0^2-3 $-8(0)^2+1$

(4) [30 points] Determine the area between the curves $y = x^2 - 3$ and $y = -8x^2 + 1$.

$$\int_{-2/3}^{2/3} -8x^2+1 - (x^2-3)\,dx$$

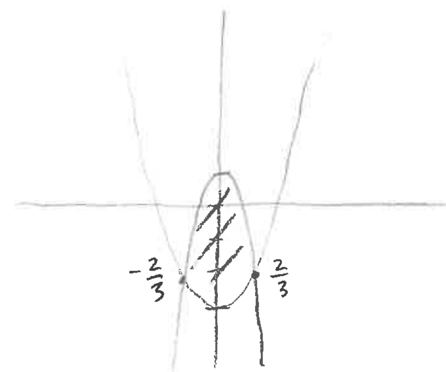
$$x^2 - 3 = -8x^2 + 1$$

$$\frac{9x^2}{9} = \frac{4}{9}$$

$$\sqrt{x^2} = \frac{\sqrt{4}}{\sqrt{9}} \qquad x = \pm\frac{2}{3}$$

$$\int_{-2/3}^{2/3} -9x^2 + 4\,dx$$

$$-3x^3 + 4x \Big|_{-2/3}^{2/3}$$



$$\left[-3\left(\tfrac{2}{3}\right)^3 + 4\left(\tfrac{2}{3}\right)\right] - \left[-3\left(\tfrac{-2}{3}\right)^3 + 4\left(\tfrac{-2}{3}\right)\right]$$

$$= \boxed{\frac{32}{9}}$$

(5) [**30 points**] Change the given rectangular equation into an equivalent polar equation.

~~$x^2 + y^2$~~ $= 81$ $r = \cancel{4}\tan x \cos x$

$$r = 4\frac{y}{\cancel{x}}\cancel{\frac{x}{r}}$$

$$r \cdot r = \frac{4y}{r} \cdot r$$

$$r^2 = 4y$$

$$x^2 + y^2 = 4y$$

$$x^2 + (y^2 - 4y) = 0$$

$$x^2 + (y-2)^2 = 4$$