CALCULUS 12 FINAL MOCK EXAM (2024)

MARKS: **Part 1 - MC** ____ **/ 22** | **Part 2 – WS** ____ **/ 58**

____ / **80**

Name KEY ______________

Date ______________

PART 1: Multiple Choice Section:

1.

If $f(x) = \frac{3}{x}$ then $f'(x) =$

A. $-\frac{3}{x^2}$ B. $\frac{3}{x}$ C. $-3x$ D. $3x$ E. *none of these*

2.

If $f(x) = \sqrt{x}$ determine the value of $f'(x)$ at $(16, 4)$

A. $-\frac{1}{4}$ B. $-\frac{1}{8}$ C. $\frac{1}{8}$ D. $\frac{1}{4}$ E. *none of these*

3.

Give all values of x where the function $f(x) = x^3 - 3x + 4$ is increasing

A. $x > 1$ B. $x < -1$ C. $-1 < x < 1$ D. $x < -1$ *or* $x > 1$

4.

If $f'(x) = -6x$ determine all values of x such that $f(x)$ is decreasing

A. $x > 0$ B. $x < 0$ C. $-6 < x < 0$ D. *all real numbers*

5.

Determine the x-values of the critical points for the function $f(x) = x^3 + 3x^2 - 24x$

A. $x = -4, \; x = 2$ B. $x = 4, \; x = -2$ C. $x = 0, \; x = 3.62, \; x = -6.62$

D. $x = 0, \; x = 3.62, \; x = 6.62$ E. *none of these*

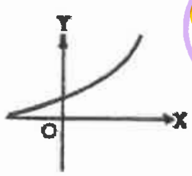
6.

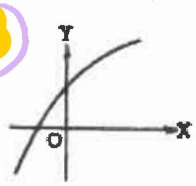
Find the slope of the tangent to $y = x^3 - 2x^2 + 6$ at $(2, 6)$

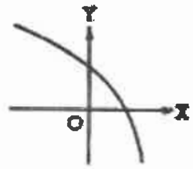
A. 4 B. 6 C. 10 D. 20 E. *none of these*

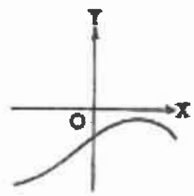
7.

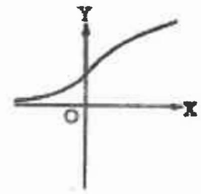
If y is a function of x such that $y' > 0$ for all x and $y'' < 0$ for all x, which of the following could be part of the graph of $f(x)$

A. 

B. 

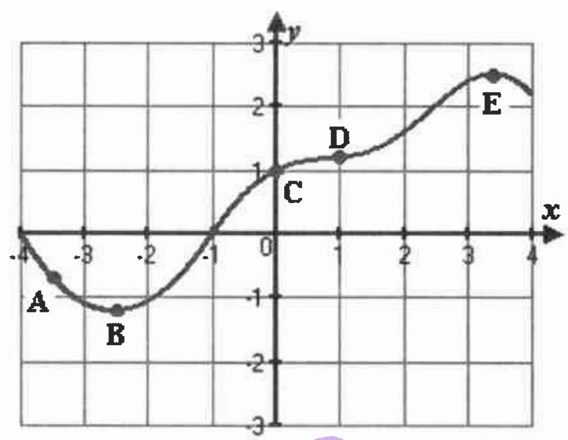
C. 

D. 

E. 

8.

At which point on the graph of $y = g(x)$ on the right is $g'(x) = 0$ and $g''(x) < 0$



A. A B. B C. C D. D E. E

9.

If $y = x(\ln x)^2$ then $\frac{dy}{dx} =$

A. $3(\ln x)^2$ B. $(\ln x)(2x + \ln x)$ C. $(\ln x)(2 + \ln x)$

D. $(\ln x)(2 + x\ln x)$ E. $(\ln x)(1 + \ln x)$

10.

Consider the curve $x + xy + 2y^2 = 6$ The slope of the line tangent to the curve at the point $(2, 1)$ is

A. $\frac{2}{3}$ B. $\frac{1}{3}$ C. $-\frac{1}{3}$ D. $-\frac{1}{5}$ E. $-\frac{3}{4}$

11.

The graph of $y = 2x^3 + 5x^2 - 6x + 7$ has a point of inflection at $x =$

A. $-\frac{5}{3}$ B. 0 C. $-\frac{5}{6}$ D. $\frac{5}{2}$ E. -2

12.

A particle moves along the x-axis so that its position at time t is $x(t) = 2t^2 - 7t + 3$ (x in cm and t in seconds). What is the velocity (in cm/sec) at time $t = 2$ seconds ?

A. -6 B. -3 C. 1 D. 4 E. ***none of these***

13.

The position function of a moving particle on the x-axis is given as $s(t) = t^3 + t^2 - 8t$ for $0 \le t \le 10$ For what values of t is the particle moving to the right ?

A. $t < -2$ B. $t > 0$ C. $t < \frac{4}{3}$ D. $0 < t < \frac{4}{3}$ E. $t > \frac{4}{3}$

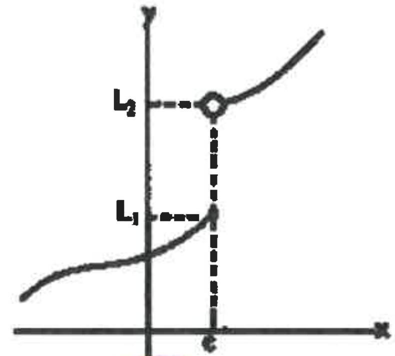
14.

A particle moves along the x-axis so that at any time t its position is given by $x(t) = te^{-2t}$ For what values of t is the particle at rest ?

A. no values B. 0 only C. $\frac{1}{2}$ only D. 1 only E. 0 and $\frac{1}{2}$

15.

Referring to the following figure showing the graph of $y = f(x)$, $\lim_{x \to c} f(x) =$



A. $\mathbf{L_1}$ B. $\mathbf{L_2}$ C. $\dfrac{\mathbf{L_1 + L_2}}{\mathbf{2}}$ D. $\mathbf{L_1 + L_2}$ E. ***does not exist***

16.

Given $f(x) = \begin{cases} x^2 & \text{where } x \neq 2 \\ 2 & \text{where } x = 2 \end{cases}$ then $\lim_{x \to 2^-} f(x) =$

A. **0** B. **1** C. **2** D. **4** E. ***does not exist***

17.

$$\lim_{x \to 0} \frac{x-1}{x^2-1} =$$

A. $-\dfrac{1}{2}$ B. $\dfrac{1}{4}$ C. $\dfrac{1}{2}$ D. **1** E. ***indeterminate***

18.

$$\lim_{x \to -3} \frac{x^2+x-6}{x^2-9} =$$

A. **−1** B. $\dfrac{5}{6}$ C. **1** D. ***does not exist*** E. ***none of these***

19.

$$\lim_{x \to \infty} \left(2x + \frac{5}{x} \right) =$$

A. 0 B. 2 C. 5 D. *does not exist* E. *infinity*

20.

$$\lim_{x \to 0} \frac{x^2 - x}{x^4 + x^3} =$$

A. -1 B. 0 C. $\frac{1}{2}$ D. 1 E. *does not exist*

21.

The approximate value of $y = \sqrt{4 + \sin(x)}$ at $x = 0.12$, obtained from the tangent to the graph at $x = 0$ is

(A) 2.00
(B) 2.03
(C) 2.06
(D) 2.12
(E) 2.24

$a = 0 \quad x = 0.12$

$L(x) = f(a) + f'(a)(x - a)$

$2 + \frac{1}{4}(0.12 - 0)$

$= 2.03$

$f(0) = \sqrt{4 + \sin 0}$

$\sqrt{4 + 0}$

$f(a) = 2$

$f'(a) = \frac{1}{2}(4 + \sin x)^{-1/2} \cos x$

$f'(0) = \frac{\cos x}{2\sqrt{4 + \sin x}}$

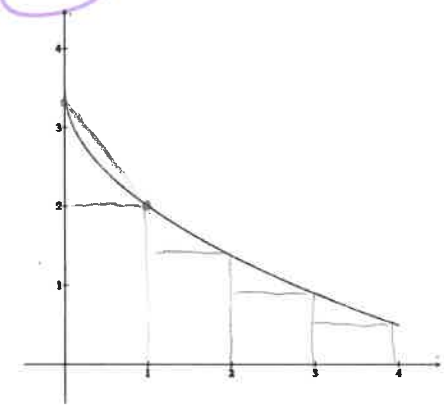
$\frac{\cos(0)}{2\sqrt{4 + \sin 0}} = \frac{1}{4}$

$f(a) = \frac{1}{4}$

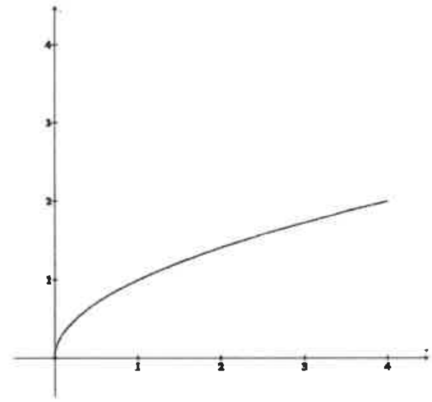
22.

If a trapezoidal sum overapproximates $\int_0^4 f(x)dx$, and a right Riemann sum underapproximates $\int_0^4 f(x)dx$, which of the following could be the graph of $y = f(x)$?

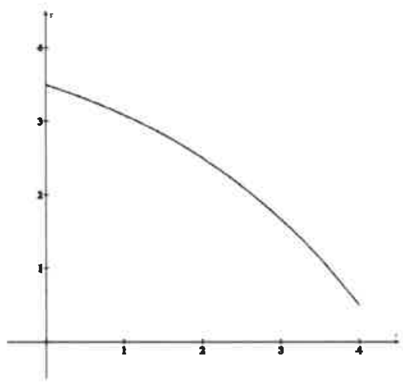
(A)



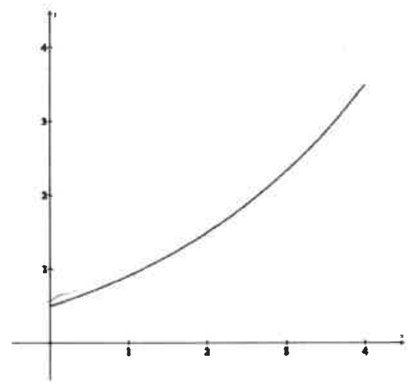
(B)



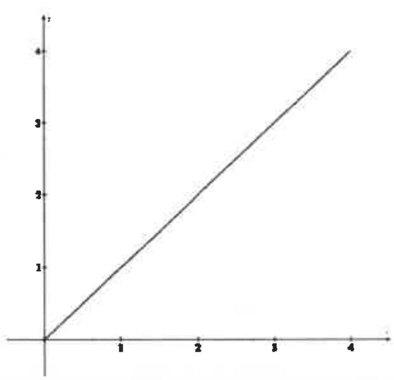
(C)



(D)



(E)



PART 2: Written Response Section:

Show all steps clearly to receive full marks. Please **CIRCLE YOUR FINAL ANSWER.**

1. Differentiate (with respect to x) each function: [1 mark each]

a) $f(x) = \frac{2}{3}x^6 - \frac{1}{2}x^4 + \frac{4}{x}$

$f'(x) = 4x^5 - 2x^3 - 4x^{-2}$

or

$4x^5 - 2x^3 - \frac{4}{x^2}$

b) $y = -2e^{x^2}$

$y' = -2e^{x^2} \cdot 2x$ or

$-4xe^{x^2}$

c) $y = cos^2(x^3 - x)$

$(\cos(x^3 - x))^2$

$2(\cos(x^3-x))^1(-\sin(x^3-x)(3x^2-1))$

or

$-2(3x^2-1)\cos(x^3-x)(\sin(x^3-x))$

d. $g(x) = \sqrt[3]{4x^2 - x - 2}$

$(4x^2 - x - 2)^{1/3}$

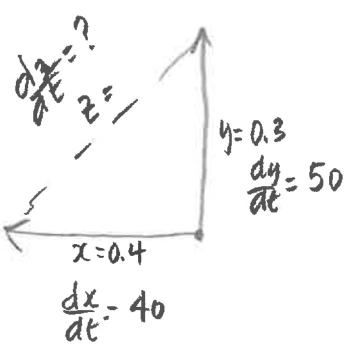
$\frac{1}{3}(4x^2-x-2)^{-2/3}(8x-1)$

or

$\frac{8x-1}{3(4x^2-x-2)^{2/3}}$

2. Related Rate Problems. Show all steps and diagram [3 marks] 62 km/hr

One car leaves an intersection traveling north at 50 *mph*, and another is driving west toward the intersection at 40 *mph*. At one point, the north-bound car is 3/10 mile north of the intersection, and the west-bound car is 4/10 mile east of the intersection. At this point, how fast is the distance between the cars changing?



$z = \sqrt{0.16 + 0.09}$ $z = 0.5$

$x^2 + y^2 = z^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

$2(0.4)(40) + 2(0.3)(50) = 2(0.5)\frac{dz}{dt}$

$32 + 30 = \frac{dz}{dt}$

52 mph

3. Find the antiderivative of each of the following functions: [2 mark each

a) $\int x^2 + \frac{5}{x} - 1 \, dx$ $\quad 5x^{-1}$

$\frac{1}{3}x^3 + 5\ln x - x + C$

b) $\int x^3(2x^4+7)^3 \, dx$

$u = 2x^4 + 7$

$du = 8x^3 \, dx$

$dx = \frac{du}{8x^3}$

$\int \cancel{x^3} u^3 \cdot \frac{du}{8\cancel{x^3}}$

$\frac{1}{8}\int u^3 \, du$

$\frac{1}{8} \cdot \frac{1}{4}u^4$

$\frac{1}{32}(2x^4+7)^4 + C$

4. Find the **exact** value of each of the following definite integrals: [2 marks each]

a) $\int_{-1}^{a} x^3 \, dx$

$\frac{1}{4}x^4 \Big|_{-1}^{a}$

$\frac{1}{4}a^4 - \frac{1}{4}$ or

$\frac{1}{4}(a^4 - 1)$

b) $\int_{0}^{6} -2e^{-2x} \, dx$

$-2e^{-2x}, \; -2$

$4e^{-2x} \Big|_0^6$

$4e^{-12} - 4e^0$

$\frac{4}{e^{12}} - 4$ or $4\left(\frac{1}{e^{12}} - 1\right)$

5. Find the **exact** area between the curve $y = x^2 - 5x + 4$ and the x-axis over the interval $0 \le x \le 4$ [2 marks]

Must find xint for BREAK-UP of $\int$

$x^2 - 5x + 4$

$(x-4)(x-1)$

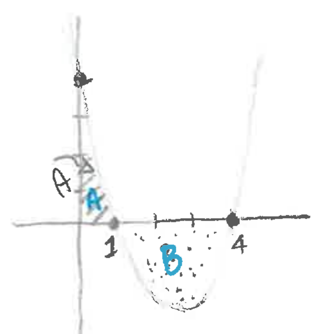
$4 \quad 1$

A $\int_0^1 x^2 - 5x + 4 \, dx \; + \; \left| \int_1^4 x^2 - 5x + 4 \, dx \right|$ B

$\frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x \Big|_0^1 + \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x \Big|_1^4$

$\frac{11}{6} + \left| -\frac{9}{2} \right|$

$\frac{11}{6} + \frac{9}{2} = \frac{19}{3}$



6. Find the **exact** area of the region bound between two functions
$f(x) = x^3 + 4x$ *and* $g(x) = 4x^2$.

[2 marks]

$\underset{T}{f(x)}$ $\quad$ $\underset{B}{g(x)}$

$x^3 + 4x = 4x^2$

$x^3 - 4x^2 + 4x = 0$

$x(x^2 - 4x + 4) = 0$

$0 \quad (x-2)(x-2) = 0$

2

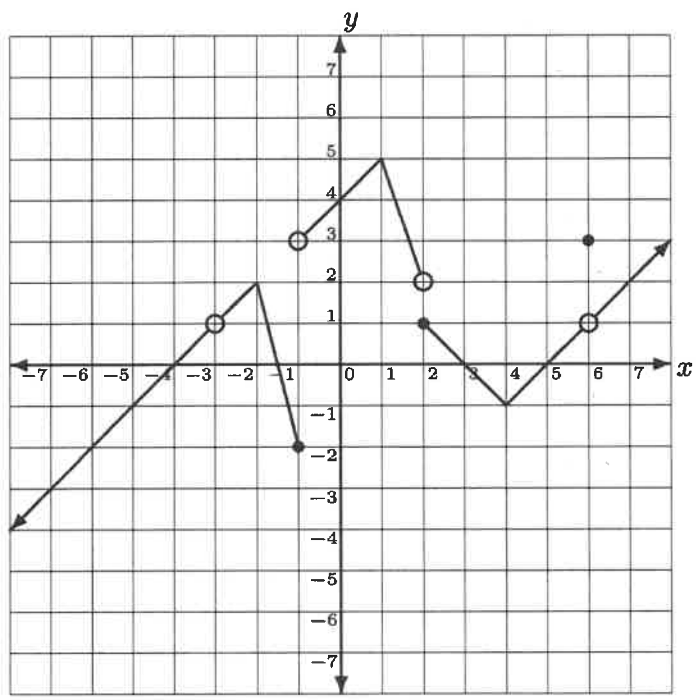
$$\int_0^2 x^3 + 4x - 4x^2 \, dx$$

$$\frac{1}{4}x^4 + 2x^2 - \frac{4}{3}x^3 \Big|_0^2$$

$$= \frac{4}{3}$$

7. [1/2 mark each]

For the function f graphed below, find the following:



$\lim_{x \to 2^-} f(x) =$ 2

$\lim_{x \to 2^+} f(x) =$ 1

$\lim_{x \to 2} f(x) =$ DNE

$f(2) =$ 1

$\lim_{x \to 6^-} f(x) =$ 1

$\lim_{x \to 6^+} f(x) =$ 1

$\lim_{x \to 6} f(x) =$ 1

$f(6) =$ 3

8. Use the **definition** of the derivative to find $\frac{dy}{dx}$ for $y = \sqrt{3x-2}$. [4 marks]

$$\lim_{h\to 0} \frac{\sqrt{3(x+h)-2} - \sqrt{3x-2}}{h}$$

$$= \frac{\left(\sqrt{3x+3h-2} - \sqrt{3x-2}\right)}{h}\frac{\left(\sqrt{3x+3h-2} + \sqrt{3x-2}\right)}{\left(\sqrt{3x+3h-2} + \sqrt{3x-2}\right)}$$

$$\lim_{h\to 0} \frac{\cancel{3x}+3h-\cancel{2}+(-\cancel{3x}+\cancel{2})}{h\left(\sqrt{3x+3h-2}+\sqrt{3x-2}\right)}$$

$$\lim_{h\to 0} \frac{3\cancel{h}}{\cancel{h}\left(\sqrt{3x+3\cancel{h}-2}+\sqrt{3x-2}\right)}$$

$$= \frac{3}{2\sqrt{3x-2}}$$

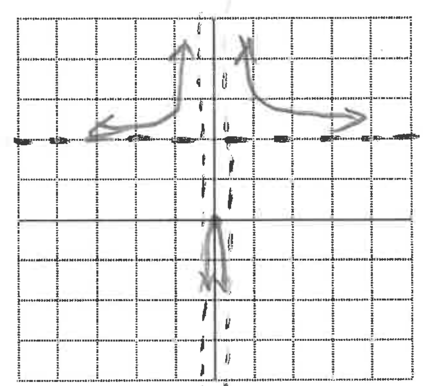
9. Make a rough sketch of the following function. Indicate x-intercepts, y-intercepts, vertical and horizontal asymptotes: [3 marks]

$$g(x) = \frac{4x^2}{2x^2-1}$$

VA: $x = \pm\frac{1}{4}$

HA: $y = 2$

x int $0,0$



$\frac{BT' - BT}{B^2}$

$$g'(x) = \frac{(2x^2-1)(8x) - (4x)(4x^2)}{(2x^2-1)^2}$$

$$= \frac{\cancel{16x^3} - 8x - \cancel{16x^3}}{(2x^2-1)^2} = \frac{-8x}{(2x^2-1)^2}$$

C.P $-8x = 0$

$(-8x)(2x^2-1)^{-2}$

F S

FS' + FS

$x = 0, 0$ → T.P $\frac{8}{-1} = -\cap$

$$g''(x) = (-8x)\left(-2(2x-1)^{-3}(4x)\right) + (-8)(2x^2-1)^{-2}$$

$$\frac{64x^2}{(2x^2-1)^3} + \frac{-8}{(2x-1)^2}\frac{(2x^2-1)}{(2x^2-1)}$$

$$\frac{64x^2 - 16x^2 + 8}{(2x^2-1)^3} = \frac{48x^2+8}{(2x^2-1)^3} = 0$$

I.P
$8(6x^2+1) = 0$
DNE

10. Use the **second derivative test** to find and classify all local extrema of $y = x^3 - 12x - 2$

[2 marks]

$y' = 3x^2 - 12 = 0$

$\frac{3x^2}{3} = \frac{12}{3}$

$\sqrt{x^2} = \sqrt{4}$

$x = \pm 2$

$y'' = 6x$

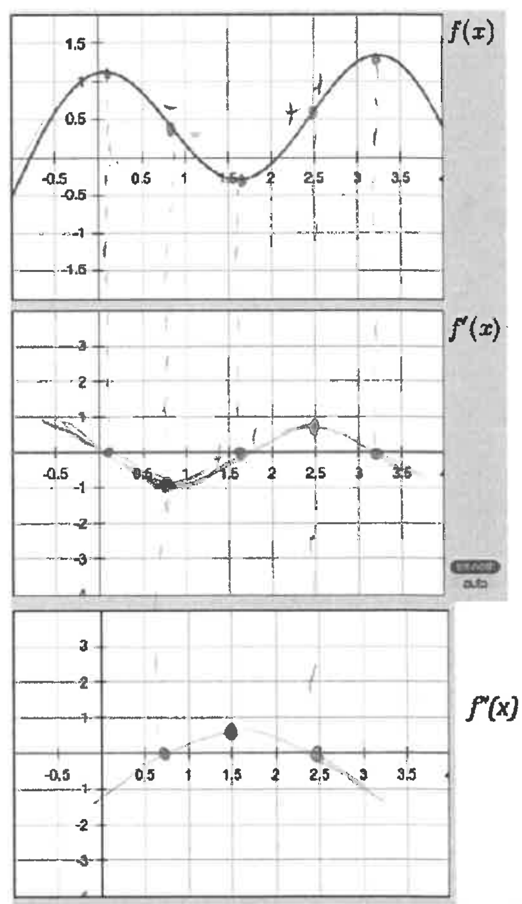
$y''(2) = 6(2) = 12$ + ∪

$y''(-2) = 6(-2) = -12$ - ∩

(2, -18) Local Min

(-2, 14) Local Max

11 a) Given the position-time graph below:
[2 marks]



i) at what time(s) is the velocity 0?

0.2 ; 1.6 ; 3.2

ii) at what time(s) is the acceleration 0?

0.75 ; 2.5

iii) when is the object speeding up?

12. Solve the following equations **exactly** *$(\ln x)^2 - 4 = 5$* [2 marks]

Let $A = \ln x$

$A^2 - 4 = 5$

$\sqrt{A^2} = \sqrt{9}$

$A = \pm 3$

$\cancel{e}^{\cancel{\ln} x} = e^3$ $\quad$ $\cancel{e}^{\cancel{\ln} x} = e^{-3}$

$x = e^3$ $\quad$ $x = e^{-3}$ or $\frac{1}{e^3}$

13. *Find the equation of the normal line to the curve $y = \ln x^2$ at the point $(e, 1)$.*
[2 marks]

$y' = \frac{1}{x^2} \cdot 2x$

$y' = \frac{2x}{x^{\cancel{2}}}$ $\quad$ $2x^{-1}$

$y' = \frac{2}{e}$

$m_T = \frac{2}{e}$ $\quad$ $m_N = -\frac{e}{2}$ $\quad$ pt $(e,1)$

$y = mx + b$

$1 = -\frac{e}{2}(e) + b$

$1 = -\frac{e^2}{2} + b$

$b = \frac{1}{1} + \frac{e^2}{2}$

$b = \frac{2+e^2}{2}$

$\therefore\ y = -\frac{e}{2}x + \frac{2+e^2}{2}$

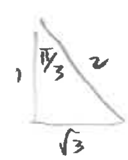
14. If $x = \sin(xy)$, find $\frac{dy}{dx}$ at the point $\left(1, \frac{\pi}{3}\right)$. [2 marks]

$1 = \cos(xy)(xy' + y)$

$1 = xy'\square + y\square$

$\frac{1 - y\square}{x\square} = \frac{\cancel{x}y'\cancel{\square}}{\cancel{x\square}}$

$y' = \frac{1 - y\cos(xy)}{x\cos(xy)}$

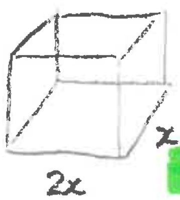


$\frac{1 - \frac{\pi}{3}\cos(\frac{\pi}{3})}{\cos(\frac{\pi}{3})} = \frac{1 - \frac{\pi}{3}\cdot\frac{1}{2}}{\frac{1}{2}} = 2\cdot\frac{1 - \frac{\pi}{6}}{\cancel{1/2}}\cdot 2 = \frac{2}{1} - \frac{\pi}{3}$

$= \frac{6-\pi}{3}$

15. *A open 3-dimensional box is to be constructed in such a way that its volume is 288 cm³. It is also specified that the length of the base is 2 times the width of the base. Determine the dimensions of the box which satisfy these conditions and have the minimum possible surface area.*

[3 marks]



z 4

6

12

$V = 2x^2 z$

$288 = 2x^2 z$

$z = \frac{288}{2x^2}$

$z = \frac{144}{x^2}$

SA =

Bottom : $2x^2$

F/B : $4xz$

Side : $2xz$

$SA = 2x^2 + 6xz$

$2x^2 + 6x \cdot \frac{144}{x^2}$

$SA = 2x^2 + 864\,x^{-1}$

$SA' = 4x - 864\,x^{-2} = 0$

$4x \cdot x^2 = \frac{864}{x^2} \cdot x^2$

$\frac{4x^3}{4} = \frac{864}{4}$

$\sqrt[3]{x^3} = \sqrt[3]{216}$

$x = 6$ $\quad z = \frac{144}{6^2} = 4$

16. Use Newton's Method to solve the following equation $f(x) = x^3 - 4x^2 + 1 = 0$ between the interval (0,1), using x-initial value $x = 0.5$. Just do two iterations $x_3 =$

[3 marks]

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$f(0.5) = 1/8$

$f'(x) = 3x^2 - 8x$

$f'(0.5) = 3(0.5)^2 - 8(0.5)$

$= -13/4$

$X_2 = 0.5 - \frac{1/8}{-13/4}$

$X_2 = 0.5384$

$f(0.5384) = -0.0034$

$f'(0.5384) = -3.43757$

$X_3 = 0.5384 - \frac{-0.0034}{-3.43757}$

$X_3 = 0.5374$

17. Evaluate the following **IMPROPER INTEGRALS** and state if convergent or divergent:

[2 marks]

$$\int_0^\infty e^{-x}\,dx$$

$e^{-x}(-1)$

$-e^{-x} \Big|_0^\infty$

$\frac{-1}{e^x} \Big|_0^\infty \qquad \frac{-1}{e^\infty} - \frac{-1}{e^0}$

$0 - \frac{-1}{1}$

$0 + 1 = 1$

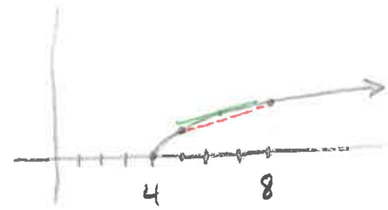
Convergent

18. Determine if the Mean Value Theorem applies to the function on the given interval. If it does, find the c-value. If it doesn't, explain why not.

[2 marks]

$f(x) = \sqrt{x-4}$ $\qquad$ $\overset{a\ \ b}{[4, 8]}$

$f'(x) = \frac{1}{2}(x-4)^{-1/2}$ $\qquad$ $f(a) = 0 \quad f(b) = 2$



$\frac{1}{2\sqrt{x-4}} = \frac{2-0}{8-4} = \frac{2}{4}$

$\frac{1}{2\sqrt{x-4}} = \frac{1}{2}$

$2 = 2\sqrt{x-4}$

$(1)^2 = (\sqrt{x-4})^2$

$1 = x - 4$

$c = 5$

19. Find the exact value of the following multiple integrals: $\int_1^3 \left(\int_{-1}^{1} 2x + 1\, dx \right) dy$

$x^2 + x \Big|_{-1}^{1}$

$1 + 1 - (-1)^2 + -1$

$\qquad\qquad 1 - 1$

$2 - 0$

$\int_1^3 2\, dy$

$2y \Big|_1^3 \Rightarrow 6 - 2 = 4$

20. Simplify: $(1+2i)(1-2i)$ $\qquad$ [1 mark]

$1 - 4i^2 \cdot {}^{-1}$

$1 + 4$

$= 5$

21. Solve the following equation over the **COMPLEX NUMBERS:** [2 marks]

$3x^2 + 5x = x - 4$

$3x^2 + 4x + 4 = 0$

$a = 3$
$b = 4$
$c = 4$

$\dfrac{-4 \pm \sqrt{4^2 - 4(3)(4)}}{6}$

$16 - 48$

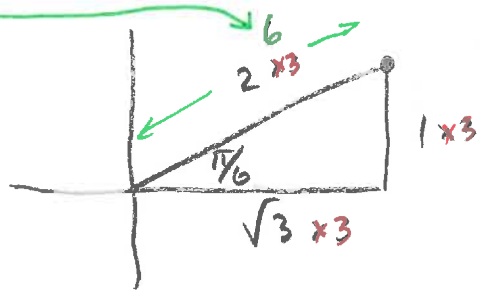
$\dfrac{-4 \pm \sqrt{-32}}{6} = \dfrac{-4 \pm 4\sqrt{2}i}{6}$

$= \dfrac{-2 \pm 2\sqrt{2}i}{3}$

22. Give the rectangular coordinates of the point whose polar coordinates are given: [2 marks]

$\left(6, \frac{\pi}{6}\right)$

$r \ \theta$



$(3\sqrt{3}, 3)$

23. Find a rectangular equation equivalent to the given polar equation and describe the graph:

[3 marks]

$r = 4cos\theta$

$r \cdot r = 4 \cdot \frac{x}{r} \cdot r$

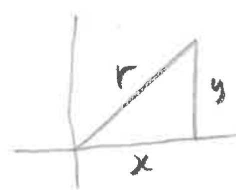
$r^2 = 4x$

$x^2 + y^2 = 4x$

$x^2 - 4x + y^2 = 0$

$(x-2)^2 + y^2 = 4$

circle $(2,0)$
radius $= 2$



$x^2 + y^2 = r^2$

$\sin\theta = \frac{y}{r}$ $\quad \cos\theta = \frac{x}{r}$