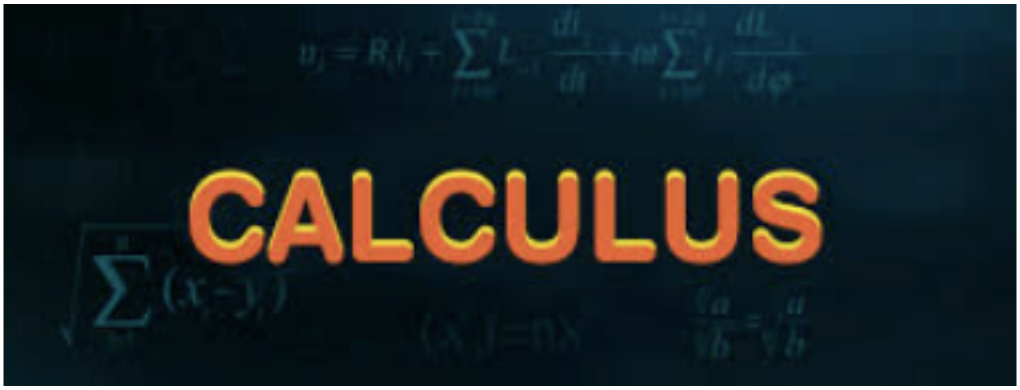


Course Introduction

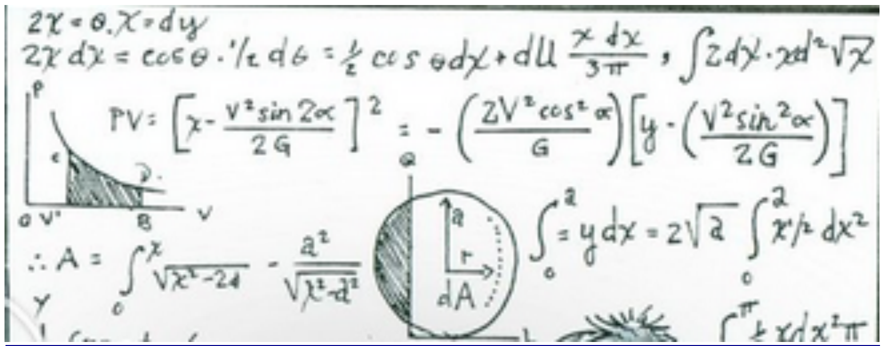
&

SEMINAR NOTES

UNIT 1

Derivatives

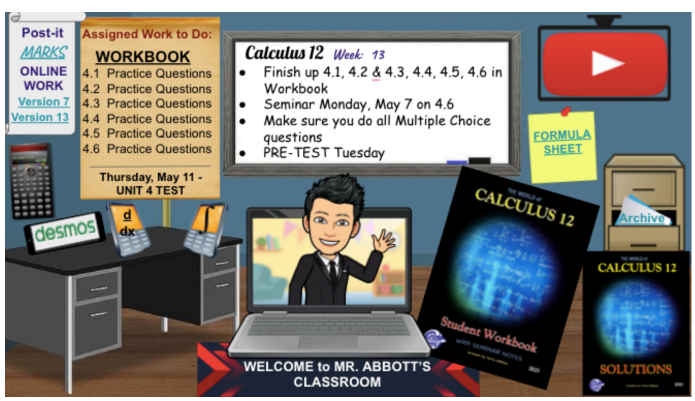
Welcome to Mr. Abbott's World of Calculus



Expectation for Success

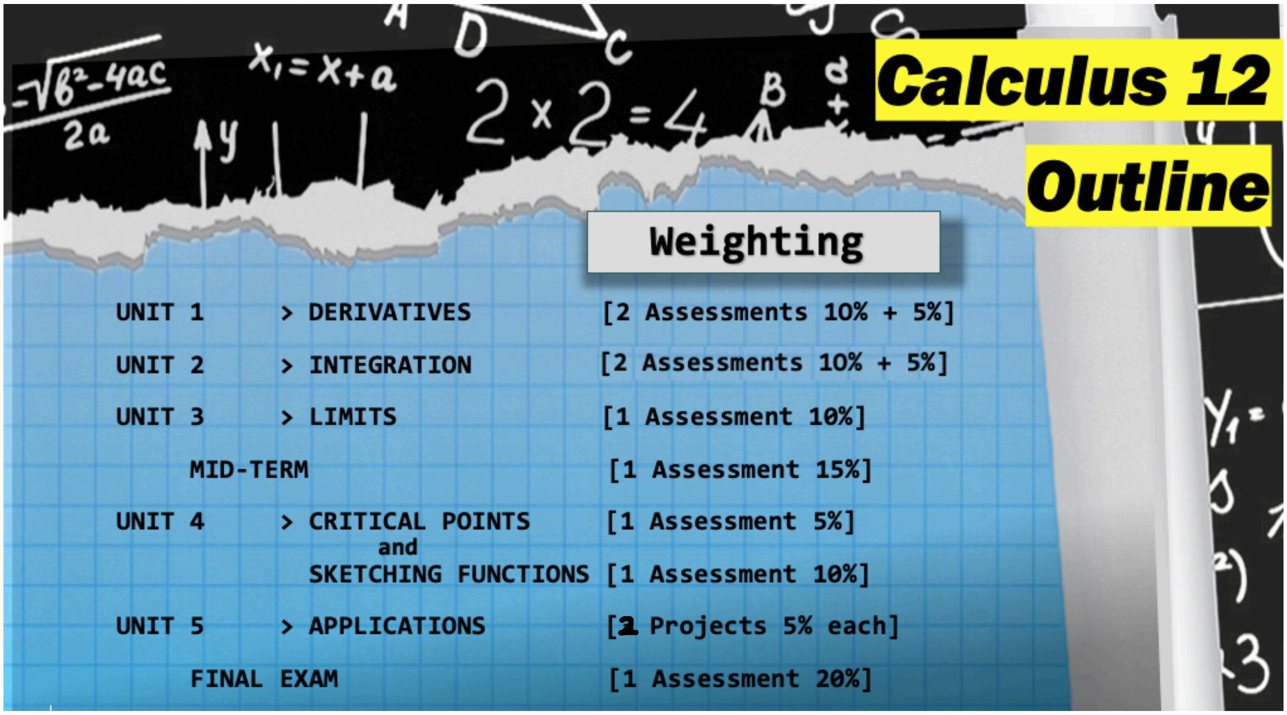
- Attend and listen carefully to Seminars
- Do book work as soon as possible after a Seminar
- Do All Worksheets and online questions that supplement the learning outcomes
- Keep on pace
- Work with a partner

Math Help >>> Before school 8:15 - 8:50 am



Calculus 12 - Virtual Classroom

https://docs.google.com/presentation/d/1aQRI5VXnzkTrZnr4x72OIQx1ahHzjffEEJp7MhLKTTU/edit#slide=id.gfd55469bbf_0_4



What is Calculus?

Go to Mr. Abbott's Virtual Classroom

Introduction & click > [Worksheet #1]

https://docs.google.com/document/d/1HYvm-frlVaD84AKaQ7hiKKIEDJIGnBNYfHCrZDlsFUM/edit

1.1 - <u>Differential Calculus</u>

[*derivatives / rate of change / slope*]

Interpretation of the Derivative

A. Rate of Change

The first interpretation of a derivative is rate of change. This is the most important interpretation of the derivative. If $f(x)$ represents a quantity at any x then the derivative $f'(a)$ represents the **<u>instantaneous</u>** rate of change of $f(x)$ at $x = a$.

Example: Suppose that the amount of water in a holding tank at t minutes is given by $V(t) = 2t^2 - 16t + 35$. Determine each of the following.

a) Is the volume of water in the tank increasing or decreasing at $t = 1$minute?

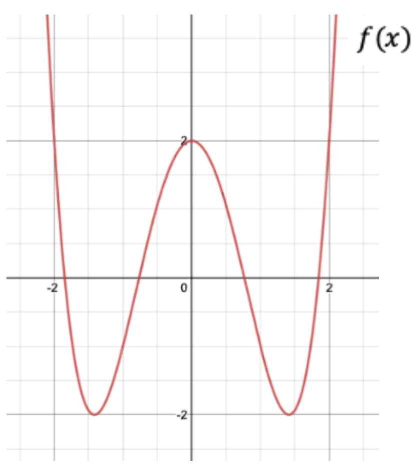
b) Is the volume of water in the tank increasing or decreasing at $t = 5$ minutes?

c) Is the volume of water in the tank changing faster at $t = 1$ or $t = 5$ minutes?

d) Is the volume of water in the tank ever not changing? If so, when?

B. Slope of Tangent Line

This is the next major interpretation of the derivative.
The slope of the tangent line to $f(x)$ at $x = a$ is $f'(a)$.



Let's say we know the equation of $f(x)$

$$f(x) = x^4 - 4x^2 + 2$$

Derivatives can give us a more accurate slope.

Example: Find the tangent line to the following function at $x = 2$.

$$f(x) = 2x^3 - 1$$

C. Velocity and Acceleration

Recall that this can be thought of as a special case of the rate of change interpretation. If the position of an object is given by $f(t)$ after t units of time the velocity of the object at $t = a$ is given by $f'(a)$. The acceleration of the object at $t = a$ is given by $f''(a)$…second derivative.

Example: An object is moving (in meters) and is given by the equation of motion $s(t) = 4t^3 - 3t^2 + 1$.

a) Find the instantaneous velocity at $t = 1$ sec.

b) Find the instantaneous acceleration at $t = 2$ sec.

The Basic Differentiation Rules

Constant > The derivative of a constant is 0: If $f(x) = c$ then $f'(x) = 0$
**(Think of the slope of a horizontal line)

Power > If $f(x) = cx$ then $f'(x) = c$
If $f(x) = x^n$ then $f'(x) = nx^{n-1}$

Power with a Constant > If $f(x) = cx^n$ then $f'(x) = n \cdot cx^{n-1}$

Product > If one function is being multiplied by another like $f(x) \cdot g(x)$

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

Quotient > If one function is being divided by another like $\frac{f(x)}{g(x)}$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

Chain > Understanding a composite function is the key to understanding the **Chain Rule**!

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

Derivative of a Constant

Examples: For each find the derivative.

$y = 24$ $\quad$ $y = 3\pi$ $\quad$ $f(x) = e^2$ $\quad$ $f(x) = \pi^{-2}$

Derivative of a Constant Generalization

$$y = C$$
$$y' =$$

Power Law

Examples: Differentiate each.

$y = x$ $\quad$ $y = x^2$ $\quad$ $f(x) = x^5$

Power Law Generalization

$$y = x^n$$
$$y' =$$

*Note: *Sometimes when using the Power Law you must "Rewrite" first, then perform* $\frac{d}{dx}$.

Examples: Differentiate each.

$y = \frac{1}{x^3}$ $\quad$ $f(x) = \sqrt[3]{x^2}$ $\quad$ $y = \frac{1}{\sqrt[5]{x^3}}$

Rewrite:

Power Law with a Constant

Examples: Differentiate each.

$y = 3x$ $\quad\quad$ $y = 5x^2$ $\quad\quad$ $f(x) = 2x^{-3}$

Power Law Constant Generalization

$$y = Cx^n$$

$$y' =$$

*Note: *Sometimes you must break down the function into parts, then perform* $\frac{d}{dx}$ *of each.*

Derivative of a Sum and Difference of Functions.

Theorem 4.32. Sum and Difference Rules. *If f and g are both differentiable functions, then*

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x),$$

and

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x).$$

Example: Find the derivative.

$$y = 5x^4 - 7x^{-3} + \frac{2}{x^3} + 3\sqrt{x^5} + \pi^3 - 2$$

Rewrite:

Symbols for Differentiating

Examples: Find the derivative of $y = x^6$

$y' = \frac{dy}{dx} =$	First Derivative
$y'' = \frac{d^2y}{dx^2} =$	Second Derivative
$y''' = \frac{d^3y}{dx^3} =$	Third Derivative
$\frac{d}{dx}\, x^6 =$	First Derivative, <u>not</u> with respect to *y*

Given the following function, differentiate accordingly.

$$y = d^4x^3 - 5m^2x^2 + 7p^4 - 24q^3 + 85$$

$\frac{dy}{dx} =$

$\frac{dy}{dd} =$ $\frac{dy}{dm} =$ $\frac{dy}{dp} =$ $\frac{dy}{dz} =$

You Try: Given the following function, differentiate accordingly.

$$y = a^6x^4 - b^5x^3 + c^4x - d^5 + 6$$

$\frac{dy}{dx} =$

$\frac{dy}{da} =$ $\frac{dy}{db} =$ $\frac{dy}{dd} =$

1.1 - PRACTICE QUESTIONS

1. *Find the first derivative of each function with respect to x:*

$a)\ y = 3x^2 - 5$ $\quad b)\ y = 8x - 2$ $\quad c)\ f(x) = 6x^2 - 3x + 2$

$d)\ y = -x^2 + 6$ $\quad e)\ g(x) = \pi x^3 + 6x - 3$ $\quad f)\ h(x) = 5\pi^2 x^4 + 6x^2 - 3\pi^4$

$g)\ k(x) = \frac{1}{4}x^8 - \frac{2}{3}x^6 + \frac{2}{5}x^4 - \frac{3}{4}$ $\quad h)\ y = 6\pi^3 - 8\pi^2 + 24$

2. *Given y, find $\frac{dy}{dx}$:*

$a)\ y = 4x^3 - 2x + 6$ $\quad b)\ y = \frac{1}{5}x^5 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + 1$

$c)\ y = x^4 - \pi^4$ $\quad d)\ y = \pi^3 x^3 - 3\pi x$

3. *Solve:*

$a)\ \text{if}\ \ p = 4q^3 + 2q^2 - 5 \qquad \text{find}\ \ \frac{dp}{dq} =$

$b)\ \text{if}\ \ g(t) = 4t^3 - 3t^2 + 6t \qquad \text{find}\ \ g'(t) =$

$c)\ \text{if}\ \ y = 2x^7 - 5x + 3 \qquad \text{find}\ \ y' =$

4. *If* $y = 2x^3 - 3x + 7$ *find:*

$a)\ \ y'\ at\ \ x = -2$

$b)\ \ y'\ at\ (1, 5)$

$c)\ \ f(0)\ and\ \ f'(0)$

5. *Find* y' *with respect to* x *if:*

$a)\ \ y = ax^3 + bx^2 + d$

$b)\ \ y = ax^4 - ax^2 + bx$

$c)\ \ y = 4ax^5 + kx^3 - Cx + D$

$d)\ \ y = D^2x^3 + 5M^3x^2 - 7$

6. *Find* $\frac{dy}{dp}$ *if:*

$a)\ \ y = 4p^3 - 2p^2 + 6$

$b)\ \ y = -5p^4 + 6p - \frac{2}{3}$

$c)\ \ y = 4mp^4 + 16p^2 - 6c$

$d)\ \ y = 6a^4 + 8a^3 - 2p^2$

7. *If* $y = 6x^5 - 2x^2 + 9x - 3$ *find:*

$a)\ y' =$

$b)\ y'' =$

$c)\ y''' =$

$d)\ \frac{dy}{dx} =$

$e)\ \frac{d^2y}{dx^2} =$

$f)\ \frac{d^3y}{dx^3} =$

$g)\ \left(\frac{dy}{dx}\right)^2 =$

$h)\ \left(\frac{d^2y}{dx^2}\right)^3 =$

$i)\ \frac{d^8y}{dx^8} =$

8. *If* $f(x)=2x^3+6x-7$ *find:*

$a)\ f'(x)$

$b)\ f''(x)$

$c)\ \left(f(x)\right)^2$

$d)\ \left(f'(x)\right)^2$

$e)\ \left(f''(x)\right)^2$

$f)\ \left(f^{(3)}(x)\right)^2$

9. *Find* $\dfrac{dy}{dx}$ *if:*

$a)\ y=Ax^3-Bx^2-C$

$b)\ y=5Ax^3-6Bx^2-Cx$

$c)\ y=5A^2x^2-6B^4x^2-C^5x$

10. *If* $y=2Dx^5-3k^2x^3+2$

$a)\ \dfrac{dy}{dx}$

$b)\ \dfrac{dy}{dD}$

$c)\ \dfrac{dy}{dk}$

$d)\ \dfrac{dy}{dz}$

11. *Simplify each expression first, and then find* y'.

$a)\ y = (2x+1)(3x-5)$ $b)\ y = (2x-3)^2$ $c)\ y = (4x)^3$

$d)\ y = x^2(x^3-6)$ $e)\ y = (\pi x)^3 - 3\pi x$ $f)\ y = \dfrac{x^2-5x+4}{x-1}$

12. *Rewrite each rational expression using exponents to remove quotients first, and then find the first derivative.*

$a)\ y = \dfrac{5}{x^2}$ $b)\ y = -\dfrac{6}{x^3}$

$c)\ y = \dfrac{2}{x^4} - \dfrac{3}{x^2} + \dfrac{5}{x} - 7x$ $d)\ y = 4x^3 - \dfrac{2}{x^2} + 7x^{-5} - \dfrac{3}{x^{-4}}$

***BE SURE TO DO MULTIPLE CHOICE QUESTIONS ONLINE **VERSION 7** #1 – 22, #48 – 55
https://moodle.sd79.bc.ca/pluginfile.php/1871/mod_resource/content/5/AB%20Calculus%20Version%207.pdf

1.2 - Derivatives using Product, Quotient & Chain Rules

Product Rule

Theorem 4.37. Product Rule. *If f and g are both differentiable functions, then*

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$

*Note: *I think of the Product Rule as F ⋅ S (First Function × Second Function)*

Abbott's Rule: $\boldsymbol{y' = FS' + F'S}$

Examples: For each find the derivative.

$y = (6x^2 - 2x)(4x^2 + 5)$

$f(x) = (5x + 3)(3x - 4x^3)$; find $f'(x)$ at $x = 1$

You Try:

$y = (x + 3x^3)(3 - x^4)$; find y′ at $x = -1$

Quotient Rule

Theorem 4.39. Quotient Rule. *If f and g are both differentiable functions, then*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

*Note: *I think of the Product Rule as* $\frac{T}{B}$ *(Top Function ÷ Bottom Function)*

Abbott's Rule:

$$y' = \frac{BT' - B'T}{B^2}$$

Example: Differentiate

$$y = \frac{2x^3}{x^4 + 5x}$$

You Try: Differentiate

$$y = \frac{x^2 + 4}{x^2 - 9}$$

*Note: *Sometimes you can switch a Quotient into a Product*

Example: Differentiate

$$y = \frac{5}{(x^2+1)^3}$$

You Try: Differentiate

$$y = \frac{3x}{(x^2-1)^2}$$

Chain Rule

Theorem 4.42. Chain Rule. *If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $h = f \circ g$ [recall $f \circ g$ is defined as $f(g(x))$] is differentiable at x and $h'(x)$ is given by:*

$$h'(x) = f'(g(x)) \cdot g'(x).$$

The **Chain Rule** computes the derivative of the ____________________ of two functions. The ____________________ $(f \circ g)(x)$ is just "g inside f" --- that is, $(f \circ g)(x) = f(g(x))$

(Note that this is *not* multiplication!)

Here are some examples:

$(x^3 - 5x + 1)^{12}$ where $g(x) =$________________ inside, and $f(x) =$_____________ outside

$\sqrt{3x+4}$ is $g(x) =$________________inside $f(x) =$________________ outside

The **Chain Rule** says that ,

In words, you differentiate the _______ function while holding the inner function fixed, then you differentiate the ______ function and ___________ them together.

In **Leibniz notation**, if $y = f(u)$ and $u = g(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

Examples: Differentiate each

$$y = (x^2 - 3x^5 + 4)^4$$

$$y = (x^2 + 1)^5(x^3 - 1)^2$$

$$f(x) = \frac{1}{\sqrt{8+x^3}}$$

You Try: Differentiate each

$y = (5x^3 - 2x^2 + 1)^6$

$f(x) = \dfrac{5x^2}{(3x+1)^4}$

$y = \left(1 + \frac{1}{\sqrt{x}}\right)^3$

In **Leibniz notation**, if $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Use the CHAIN RULE to find $\frac{dy}{dx}$ *at the indicated value x:*

Example: $y = \sqrt{u^2 + 3}$ $\quad u = 2x^2 - 1$ $\quad x = 1$

Note: The chain can have any number of links. Eg. $y = f(g(h(x)))$

So , if $y = f(u)$, $u = g(v)$ and $v = h(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$

Use the CHAIN RULE to find $\frac{dy}{dx}$ *at the indicated value x:*

$y = \sqrt{2 + u}$ $\quad u = \sqrt{2 + v}$ $\quad v = \sqrt{2 + x}$ $\quad x = 2$

1.2 - PRACTICE QUESTIONS

1. *Use the **Product Rule** to find the first derivative.*

$a)\ y = (3x + 1)(2x - 5)$

$b)\ y = 3x^2(8x - 3)$

$c)\ y = (2x + 1)(4x^2 - 4x + 1)$

$d)\ y = (3x^3 - 2x^2)(3x^3 + 2x^2)$

2. *Find* $\frac{dy}{dx}$ *at the given value of* x.

$a)\ y = (2 + 7x)(x - 3);\ x = 2$

$b)\ y = (1 + 2x)(1 - 2x);\ x = \frac{1}{2}$

3. *Use the **Quotient Rule** to find the first derivative.*

$a)\ y = \dfrac{x^2}{2x+1}$

$b)\ y = \dfrac{4x^2}{1-6x^3}$

$c)\ y = \dfrac{x^2-4x}{x+2}$

$d)\ y = \dfrac{x^2-9}{x^2+9}$

$e)\ y = \dfrac{x^3}{8-x^3}$

$f)\ y = \dfrac{4-x^2}{3x}$

4. *Find* $\dfrac{dy}{dx}$ *at the given value of* x.

$a)\ \ y = \dfrac{x+1}{2x^2-1}$, $x = 0$

$b)\ \ y = \dfrac{x^2-1}{x^2+1}$, $x = 1$

$c)\ \ y = \dfrac{x^3}{8-x^3}$, $x = -1$

$d)\ \ y = \dfrac{2+x^2}{3x}$, $x = -2$

5. *Use the **Chain Rule** to find the first derivative.*

$a)\ y = (6x^2)^5$

$b)\ y = (-3x^4)^5 + 6x^2 - 7x$

$c)\ y = (p^2 - 3p + 1)^4$

$d)\ y = (x^2 - 1)^3(2x - 1)^4$

$e)\ y = \dfrac{6x}{(x^2 + 1)^4}$

$f)\ y = (2x^4 + 8)^{\frac{1}{2}}$

$g)\ y = (3t^4 - 2t)^{\frac{1}{4}}$

$h)\ y = \sqrt{5x + 7}$

6. *Use the **Chain Rule** to find the first derivative.* **Continued

$i)\ y = \dfrac{1}{\sqrt{4+t^2}}$

$j)\ y = \left(1 + u^{\frac{1}{3}}\right)^6$

$k)\ y = \left(1 + \sqrt[3]{u}\right)^6$

$l)\ y = \left(1 + \dfrac{1}{\sqrt[3]{x}}\right)^6$

$m)\ y = (\pi x)^3 + 2\pi^2 x + 6\pi x$

$n)\ y = (2x^3 + x)^4$

$o)\ y = \dfrac{6\pi x}{(x^3 - \pi)^2}$

$p)\ y = 4x^2(2x - 5)^3$

7. *Find the first derivative of each expression below.*

$a)\ y = \pi x + (5\pi x)^3$

$b)\ y = (1 - x + 2x^2 - 3x^3)^4$

$c)\ y = ((2x)^4 + (16 - x)^3)^2$

$d)\ y = \dfrac{(2x-1)^2}{(x-2)^3}$

$e)\ y = (2x - 1)^{-3}$

$f)\ y = \dfrac{\pi x}{(x^3 - \pi)^2}$

$g)\ y = \sqrt{x}(1 - 2x)^5$

$h)\ y = \left(\dfrac{x^2 - 1}{x^2 + 1}\right)^2$

8. Use the CHAIN RULE to find $\frac{dy}{dx}$ at the indicated value x:

a) $y = 2u^2 + 5$ $\quad$ $u = 3x$ $\quad$ $x = 1$

b) $y = \frac{5}{u+2}$ $\quad$ $u = 3x - 2$ $\quad$ $x = 1$

c) $y = \sqrt{u^2 + 3}$ $\quad$ $u = 2x^2 - 1$ $\quad$ $x = 1$

d) $y = 2u^2$ $\quad$ $u = 3v$ $\quad$ $v = 2x + 1$ $\quad$ $x = 0$

e) $y = 4u^3 - 3u^2$ $\quad$ $u = 2v^2 + 4v$ $\quad$ $v = 1 - 2x^2$ $\quad$ $x = -1$

EXTENDED QUESTIONS

Find the derivative of each of the following functions and simplify.

1. $f(x) = 4x^3 - 3x^2 + 2x - \pi$

2. $f(x) = \dfrac{x^2}{3} - \dfrac{3}{x^2}$

3. $f(x) = -3\left(2x^2 - 5x + 1\right)$

4. $f(x) = \sqrt{x} - \dfrac{1}{\sqrt{x}}$

5. $f(x) = \dfrac{x+1}{x-2}$

6. $f(x) = \dfrac{x^2 - 2}{x^2}$

7. $f(x) = \dfrac{x^2}{x^2 - 2}$

8. $f(x) = \sqrt{x}\left(x^2 + 1\right)$

10. $f(x) = \dfrac{2}{\sqrt{x}} + \dfrac{\sqrt{x}}{2}$

11. $f(x) = \dfrac{2x}{x-1}$

12. $f(x) = (3x-2)(2x+1)$

13. $y = 5x^2 - 5\sqrt{x} - \dfrac{3}{x}$

14. $y = \dfrac{\sqrt{x}}{\sqrt{x}-1}$

16. $y = 6x^{\frac{-3}{2}} + 7x^{\frac{1}{5}} + 1$

17. $y = \dfrac{-7}{1-x^3}$

18. $y = \dfrac{4}{3}x^{\left(\frac{3}{4}-\pi\right)}$

19. $y = \dfrac{1}{7x}$

1.3 - "Simplifying Completely" using Chain Rule

$$y=\frac{6x}{\left(x^2+1\right)^4}$$

$$y=\frac{(x+4)^{\frac{1}{2}}}{(x-4)^{\frac{1}{2}}}$$

1.3 - PRACTICE QUESTIONS

Find the derivative for each of the following and simplify.

1. $g(t)=\dfrac{(t+3)^4}{\left(t^2+5\right)^{\frac{1}{2}}}$

2. $f(x)=x^4(5x-1)^3$

3. $y = x^2\sqrt{x^3+1}$

4. $f(x) = \dfrac{4x^4 - 4x^2 + 5}{2x^{5/3} + 3}$

Find $f'(x)$ and $f''(x)$, simplified for the following.

1. $f(x) = \dfrac{1}{x^2 + 1}$

2. $f(x) = \sqrt{4x - x^2}$

1.4 - Position, Velocity & Acceleration Functions

- ***Position Function is first derivative***
- ***Velocity Function is second derivative***
- ***Acceleration Function is third derivative***

a) **Position**

· The particle can move in either direction and is denoted by _______ where ____ represents the position of the particle on the line at time *t*.

b) **Velocity**

· rate of change of ____ measured in m/s or ft/s

c) **Acceleration**

· rate of change of ____ measured in m/s^2 or ft/s^2 (how quickly a particle picks up or loses speed)

Example: Find the velocity and acceleration as functions of time (*t*).

$a)\ s(t) = 6t^4 + 2t^2 - t + 9$ $\qquad$ $b)\ s(t) = 2t\left(5t^2 - 1\right)^2$

1.4 - PRACTICE QUESTIONS

1. *Each position function below describes motion in a straight line. Find the velocity and acceleration as functions of time* (t).

$a)\ s(t) = 5t^2 - 2t + 7$

$b)\ s(t) = 4t^4 - \frac{1}{2}t^2 + 3$

$c)\ s(t) = 6t - 8$

$d)\ s(t) = t - 8 + \frac{6}{t}$

$e)\ s(t) = t(t-3)^2$

$f)\ s(t) = t + \frac{4t}{t+2}$

1.5 - Implicit Differentiation

To this point we've done quite a few derivatives, but they have all been derivatives of functions of the form $y = f(x)$. Unfortunately, not all the functions that we're going to look at will fall into this form.

Example: Find y' for $xy = 1$

Now, recall that we have the following notational way of writing the derivative.

$$\frac{d}{dx}(y(x)) = \frac{dy}{dx} = y'$$

Note: For most of the exercise questions we will use $\boldsymbol{y'}$.

Using the above notation let's find the derivative for the following.

Given	Derivative
y	
y^2	
y^3	
y^4	

Example 1: Find y' for $xy = 1$

Example 2: Find y' for $4y^2 - 6y^3 + 2x = 4x^2$

Example 3: Find y' for $x^2y^3 = 8x$

You Try: Find y' for:

a) $5x^2 + 6y^4 - 3y = 5x$

b) $x^3y^2 = 4x^2$

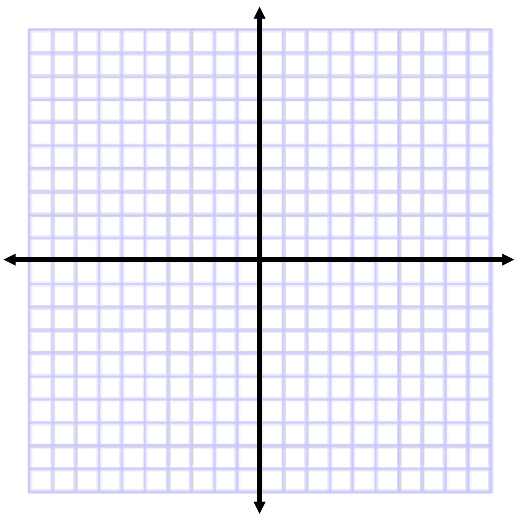
Implicit Differentiation Problems

Differentiating implicitly is important when we deal with "Related Rates".

Let's look at two different situations.

Example 1:

$x^2 + y^2 = 100 \quad find \ \frac{dy}{dx} \ at \ x = 6$



Example 2:

$If \ x^2 + y^2 = 12 \ and \ \frac{dx}{dt} = 5, \ find \ \frac{dy}{dt} \ at \ (3, 7).$

You Try:

Given: $x^2 + y^2 = z^2$; find $\frac{dz}{dt}$ if $x = 1,\ y = -3,$ $\frac{dx}{dt} = 2$, and $\frac{dy}{dt} = 1$.

1.5 - PRACTICE QUESTIONS

1. *Use* ***<u>IMPLICIT DIFFERENTIATION</u>*** *to find* $\frac{dy}{dx}$ *in terms of x and y.*

$a)\ 4x^2 + y^2 = 8$

$b)\ 3x - 4y^2 = 2$

$c)\ x^2 + 5y^2 + y = 10$

$d)\ xy^2 = 4$

$e)\ x^2 + 2xy - y^2 = 13$

$f)\ y^3 + y = 4x$

$g)\ y(x^2 + 3) = y^4 + 1$

$h)\ xy^3 + x^3y = 2$

2. *If* $x^2 + y^2 = 8$ *and* $\frac{dx}{dt} = 3$, *find* $\frac{dy}{dt}$ at (-2, 2).

3. *If* $x^2 + y^2 = z^2$ *and* $\frac{dx}{dt} = -2$, $\frac{dy}{dt} = -1$, $x = 1$ *and* $y = -3$, *find* $\frac{dz}{dt}$.

***BE SURE TO DO MULTIPLE CHOICE QUESTIONS ONLINE #179 – 189
https://moodle.sd79.bc.ca/pluginfile.php/1871/mod_resource/content/5/AB%20Calculus%20Version%207.pdf

EXTENDED QUESTIONS

For each problem, use implicit differentiation to find $\frac{dy}{dx}$ in terms of x and y.

1) $2x^3 = 2y^2 + 5$

2) $3x^2 + 3y^2 = 2$

3) $5y^2 = 2x^3 - 5y$

4) $4x^2 = 2y^3 + 4y$

5) $5x^3 = -3xy + 2$

6) $1 = 3x + 2x^2y^2$

7) $3x^2y^2 = 4x^2 - 4xy$

8) $5x^3 + xy^2 = 5x^3y^3$

9) $2x^3 = (3xy + 1)^2$

10) $x^2 = (4x^2y^3 + 1)^2$

11) $\sin 2x^2y^3 = 3x^3 + 1$

12) $3x^2 + 3 = \ln 5xy^2$

For each problem, use implicit differentiation to find $\frac{d^2y}{dx^2}$ in terms of x and y.

13) $4y^2 + 2 = 3x^2$

14) $5 = 4x^2 + 5y^2$

1.6 - Related Rates

Example: The area of a circular gas leak on the frozen tundra of Northern Alberta is increasing at the rate of 80 m^2/s. How fast is the radius changing when:

a) the radius is 30 m.

b) the area is 800 m^2.

You Try: How fast is the side of a square shrinking when the side is 10m and the area is decreasing at a rate of 5m^2/sec?

Example: A snowball rolling down the hill increases in size in such a way that the instant at which its radius is 20 cm, its radius is increasing at 9 cm/min. At what rate is the volume of the snowball changing at that instant?

1.6 - PRACTICE QUESTIONS

1. *If A is the area of a circle of radius r, find* $\frac{dA}{dt}$ *in terms of* $\frac{dr}{dt}$.

2. *The area of a circular oil slick on the surface of the sea is increasing at the rate of* $150\ m^2/s$. *How fast is the radius changing when:*

   a) *the radius* $25\ m$.

   b) *the area is* $1000\ m^2$

3. *How fast is the side of a square shrinking when the length of the side is* 2 m *and the area is decreasing at* $0.25\ m^2/s$?.

4. *The hypotenuse of a right triangle is of fixed length but the lengths of the other two sides* x *and* y *depend on time. How fast is y changing when* $\frac{dx}{dt} = 4$ *and* $x = 8$ *if the length of the hypotenuse is* 17?

5. *A spherical balloon is inflated so that the volume is increasing at the rate of* $5\ m^3/min$.

a) *at what rate is the diameter increasing when the radius is* $6\ m$?

b) *at what rate is the diameter increasing when the volume is* $36\ m^3$?

6. *Two cars approach an intersection, one traveling east and the other north. If both cars are traveling at* $70\ km/h$, *how fast are they approaching each other when they are both* $0.5\ km$ *from the intersection?*

EXTENDED QUESTIONS

1) A spherical snowball melts in such a way that the instant at which its radius is 20 cm, its radius is decreasing at 3 cm/min. At what rate is the volume of the ball of snow changing at that instant?

2) A spherical snowball is melting. Its radius decreases at a constant rate of 2 cm per minute from an initial value of 70 cm. How fast is the volume decreasing half an hour later?

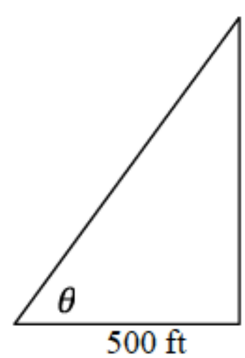
3) A 3 meter ladder stands against a wall. The foot of the ladder moves outward at a speed of .1 meters/sec. when the foot is 1 meter from the wall. At that moment, how fast is the top of the ladder falling? What if the foot has been 2 meters from the wall?

4) An airplane flying at 450 km/hr at a constant altitude of 5 km, is approaching a camera mounted on the ground. Let θ be the angle of elevation above the ground at which the camera is pointed. When $\theta = \pi / 3$, how fast does the camera have to rotate in order to keep the plane in view?

5) Two cars start moving from the same point. One travels south at 60 mph and the other travels west at 25 mph. At what rate is the distance between the cars increasing two hours later?

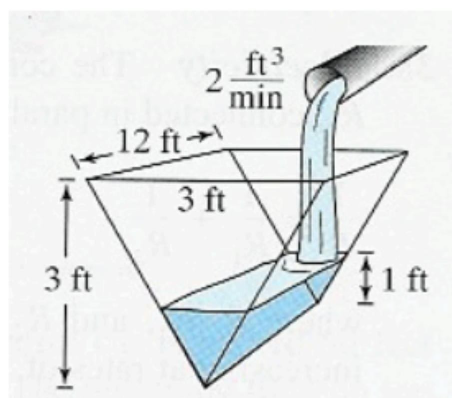
6) Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?

8. A hot-air balloon rising straight up from a level field is tracked by a range finder 500 ft from the lift-off point. At the moment the range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?



9. Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10ft and a base radius of 5ft. How fast is the water level rising when the water is 6ft deep?

10. A trough 3ft wide and 12ft long is being filled at a rate of 2 cubic feet per minute. The ends of the trough are isosceles triangles with altitudes 3ft. How fast is the water level rising when the depth is 1ft?



1.7 - Trigonometric Functions

Theorem 4.54. Derivatives of Basic Trigonometric Functions.

$$\frac{d}{dx}(\sin(x)) = \cos(x) \qquad \frac{d}{dx}(\cos(x)) = -\sin(x)$$
$$\frac{d}{dx}(\tan(x)) = \sec^2(x) \qquad \frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$$
$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x) \qquad \frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

Examples: Find the derivative of the following.

$y = \sin(x^3 + 5x)$

$y = \dfrac{\sin(2x)}{\cos^2(x)}$

$y = \sin^2(x^3 + 5x)$

$y = 5\sec(3x^2)$

$y = x^2 \sin(x)$

$y = \left(\sin(x^3 + 5x)\right)^2$

You Try: Find the derivative of the following.

$y = \sin^4(3x^2 - 5x)$

$y = 8\csc(4x^3)$

$y = x^3\cos(x)$

1.7 - PRACTICE QUESTIONS

1. *Find* $\frac{dy}{dx}$:

$a)\ y = 3\cos(4x)$

$b)\ y = cos\left(3x + \frac{\pi}{2}\right)$

$c)\ y = cos(2x^3)$

$d)\ y = cos^3(2x)$

1. *Continued - Find* $\frac{dy}{dx}$:

$e)\ y = cos(x^2 + x)$

$f)\ y = (x + \cos(x))^2$

$g)\ y = 2sin(\pi x) + x^2$

$h)\ y = 3sin(x^2 - 1)$

$i)\ y = (\sin(2x) + \cos(x))^2$

2. *Differentiate each function:*

$a)\ f(x) = x\cos(x)$

$b)\ g(x) = x^3 \sin(2x)$

2. *Continued - Differentiate each function:*

$c)\ k(x) = x^3 cos(3x^2)$

$d)\ h(x) = \sin\left(\cos(\pi x)\right)$

$e)\ m(x) = \sin(x)\cos(x)$

$f)\ p(x) = \dfrac{\sin(2x)}{cos(2x)}$

$g)\ g(x) = sin^2\left(x^{\frac{1}{2}}\right)$

$h)\ k(x) = \sin\left(\dfrac{1}{x}\right)$

3. *Find* $\dfrac{dy}{dx}$ *in each case where A, B, m and n are constants:*

$a)\ y = cos(Ax + B)$

$b)\ y = cos^n(Bx)$

$c)\ y = sin^m(x^n)$

$d)\ y = Ax^n sin^m(Bx)$

4. *Find* $\frac{dy}{dx}$ *in each case:*

$a)\ y = \tan(x) - \tan\,(2x)$

$b)\ y = 3sec(2x^2 + 1)$

$c)\ y = \frac{x^2}{\tan\,(x)}$

$d)\ y = tan(x^2) - tan^2(x)$

$e)\ y = x^2 tan\left(\frac{1}{x}\right)$

$f)\ y = sin\big(tan(x^3)\big)$

5. *Find* $\frac{dy}{dx}$ *in each case. Watch for the need for Implicit Differentiation!*

$a)\ y = \cot(2x) + \csc(2x)$

$b)\ y = 2x^3 \cot(x)$

$c)\ y = (x + \csc(x))^2$

$d)\ y = \sqrt{\pi^2 + csc^2(x)}$

$e)\ y = \dfrac{\cot(x)}{1 + csc^2(x)}$

$f)\ y = \sqrt{x}\, \csc(x)$

$g)\ y = \sin(\mathrm{xy})$

$h)\ y = \cot(x + y)$

1.8 - ln(x) & e^x Functions

Theorem 4.58. Derivative Formulas for e^x and $\ln x$.

$$\frac{d}{dx}e^x = e^x \text{and} \frac{d}{dx}\ln x = \frac{1}{x}.$$

Examples: Find the derivative of the following.

$y = e^x$

$y = lnx$

$y = e^{3x}$

$y = \ln(x^4)$

$y = x^2e^{4x}$

$y = \ln(x^3 + 5x)$

$y = 3xe^{sin(x^2)}$

$y = x^2\ln(x^4)$

1.8 - PRACTICE QUESTIONS

1. *Differentiate each function:*

$a)\ f(x) = 5e^{2x}$ $\quad b)\ h(x) = 2e^{x^2 - x}$ $\quad c)\ k(x) = 3e^{2\sin x}$

$d)\ p(x) = x^2 e^x$ $\quad e)\ q(x) = x^2 e^{-3x}$ $\quad f)\ m(x) = \left(e^{2x} - e^{-2x}\right)^2$

$g)\ g(x) = \sqrt{x}\ e^{\sqrt{x}}$ $\quad h)\ f(x) = \ln\left(\pi + e^{2x}\right)$ $\quad i)\ m(x) = \dfrac{e^{2x}}{1 + e^{2x}}$

$j)\ g(x) = e^x \ln\sqrt{x}$ $\quad k)\ w(x) = \ln\left(e^x + e^{-x}\right)$ $\quad l)\ r(x) = \dfrac{e^x}{\ln x}$

2. *If y defined implicitly as a function of x by the given equation, find* $\frac{dy}{dx}$:

$a)\ \ x + y\ln x = 2$ $\qquad$ $b)\ \ y - e^{xy} = 5$ $\qquad$ $c)\ \ e^{\sin 2y} + 2x = 4y$

3. *Find* $\frac{dy}{dx}$:

$a)\ \ y = \ln\left|x^2 - 1\right|$ $\qquad$ $b)\ \ y = \ln\left|x^3 - 7x + 1\right|$ $\qquad$ $c)\ \ y = \left(\ln|x|\right)^3$

$d)\ \ y = \ln|\tan x|$ $\qquad$ $e)\ \ y = \cos x \ln|\cos x|$ $\qquad$ $f)\ \ y = \sin\left(\ln|x|\right)$

***BE SURE TO DO MULTIPLE CHOICE QUESTIONS ONLINE **VERSION 7**.
#96, 98, 99, 100, 101,103, 105, 106, 109, 122, 141, 142, 143, 145
https://moodle.sd79.bc.ca/pluginfile.php/1871/mod_resource/content/5/AB%20Calculus%20Version%207.pdf

1.9 - Exponential & Logarithmic Functions

Theorem 4.59. Derivative Formulas for a^x and $\log_a x$.

$$\frac{d}{dx}a^x = (\ln a)a^x \text{ and } \frac{d}{dx}\log_a x = \frac{1}{x\ln a}.$$

Examples: Find the derivative of the following.

$y = 2^x$

$y = 5^{x^2+7x}$

*Note: *Let's review the Logarithmic Properties before diving onto derivatives of logs.*

Logarithmic Properties	
Product Rule	$\log_a(xy) = \log_a x + \log_a y$
Quotient Rule	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
Power Rule	$\log_a x^p = p\log_a x$
Change of Base Rule	$\log_a x = \frac{\log_b x}{\log_b a}$
Equality Rule	If $\log_a x = \log_a y$ then $x = y$

Base = Argument	$\log_b b = 1$
Change of Base.1	$\log_8 2^x =$
Change of Base.2	$16^{\log_4 8} =$

Write expression in terms of individual logarithms of x, y, & z. $\log_7 \frac{x^2}{y^3\sqrt{z}}$

Write each expression as a single logarithm in reduced form. $\log A + 2\log B - \frac{1}{3}\log c$

Log Equations:

$\log(x+2) + \log(x-1) = 1$

$\log_2 x + \log_2(x-2) = \log_2 8$

Theorem 4.59. Derivative Formulas for a^x and $\log_a x$.

$$\frac{d}{dx}a^x = (\ln a)a^x \text{ and } \frac{d}{dx}\log_a x = \frac{1}{x\ln a}.$$

$y = \log_b a$

$y' = \dfrac{a'}{a \ln b}$

Examples: Find the derivative of the following.

$y = \log_2 x$

$y = \log_5(3x^2)$

$y = [\log_4(x^2 - 3x)]^2$

$y = \log_5(4xe^x)$

You Try: Find the derivative of the following.

$y = \log_4(x^2 - 3x)$

$y = \log(2 + 3x)$

1.9 - PRACTICE QUESTIONS

1. *Differentiate each function:*

$a)\ \ y = 2^x$

$b)\ \ y = 10^{x^3}$

$c)\ \ y = 2^{sinx}$

$d)\ \ y = \pi^{x^2}$

$e)\ \ y = 3^{x^2+3x}$

$f)\ \ y = (2x^3)(3^{2x})$

2. Differentiate each function with respect to x.

a) $y = log_3\,(3x^2)$

b) $y = log_2\,(4x^2)$

c) $y = [log_3\,(3x^5 + 5)]^5$

d) $y = log_5\,(-5x^3 - 2)^2$

1.10 - Working with Numerical Values

Examples: Suppose that functions *f* and *g* and their derivatives have the following values at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	**8**	**2**	**1/3**	**-3**
3	**3**	**-4**	2π	**5**

$a)\ 2f(x)\ at\ \ x = 2$

$b)\ f(x) + g(x)\ at\ \ x = 3$

$c)\ f(x) \cdot g(x)\ at\ \ x = 3$

$d)\ \frac{f(x)}{g(x)}\ at\ \ x = 2$

$e)\ f\big(g(x)\big)\ at\ \ x = 2$

1.10 - PRACTICE QUESTIONS

1. *Two functions, $f(x)$ and $g(x)$, are continuous and differentiable for all real numbers. Some values of the functions and their derivatives are given in the following table.*

x	0	1	2	3	4
$f(x)$	$\frac{1}{2}$	$\frac{1}{3}$	1	-1	3
$g(x)$	-2	1	$-\frac{1}{2}$	2	$-\frac{1}{3}$
$f'(x)$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{1}{4}$	0	$-\frac{4}{5}$
$g'(x)$	-1	$\frac{2}{3}$	-4	-3	$-\frac{1}{3}$

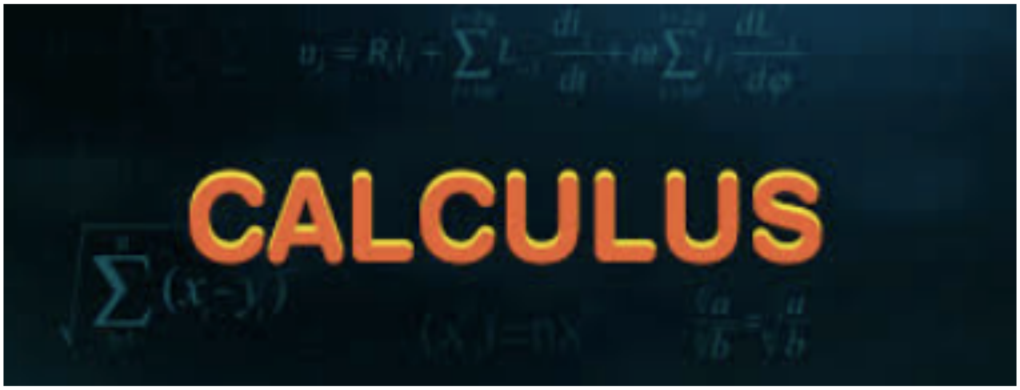
Based on the table, calculate the following:

$a)\ \frac{d}{dx}(f(x) + g(x)), evaluated\ at\ x = 4$

$b)\ \frac{d}{dx}(f(x)g(x)), evaluated\ at\ x = 1$

$c)\ \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right), evaluated\ at\ x = 0$

$d)\ \frac{d}{dx}(f(g(x))), evaluated\ at\ x = 3$



SEMINAR NOTES

UNIT 2

2.1 - Antiderivatives

1. Antiderivative or Indefinite Integral

The antiderivative is represented by ______ whose derivative is $f(x)$.

The purpose of finding antiderivatives is called antidifferentiation or ____________________. To find an antiderivative, we go backwards to find the original function.

> **Definition**
>
> Given a function $y = f(x)$, any function $y = F(x)$ such that
>
> $$F'(x) = f(x)$$
>
> is called an ***antiderivative*** of $f(x)$.

Example 1

$\int$ notation you'll see for integration

Find Derivative	Find Antiderivative
$\frac{d}{dx}[5] =$	$\int$ ____ $dx =$
$\frac{d}{dx}[2x] =$	$\int$ ____ $dx =$
$\frac{d}{dx}[2e^x] =$	$\int$ ____ $dx =$
$\frac{d}{dx}[5x^3] =$	$\int$ ____ $dx =$

General Derivative Rule: **General Antiderivative Rule:**

Function	Particular antiderivative		Function	Particular antiderivative
$cf(x)$	$cF(x)$		$\cos x$	$\sin x$
$f(x)+g(x)$	$F(x)+G(x)$		$\sin x$	$-\cos x$
$x^n \quad (n \neq -1)$	$\frac{x^{n+1}}{n+1}$		$\sec^2 x$	$\tan x$
			$\sec x \tan x$	$\sec x$

Another way to find Antiderivative

If $\frac{dy}{dx} = 3x^2 + 5x - 2$, then find $y =$

Another way to find Antiderivative is called:

Integrals or *Integration* $\int$ notation you'll see for integration

INTEGRALS

Indefinite $\int x^2\,dx$ **Definite** $\int_1^5 x^2\,dx$

DEFINITION OF INDEFINITE INTEGRAL

Given a function, *f(x)*, an antiderivative of *f(x)* is any function F*(x)* such that F'*(x)* = *f(x)*

If F*(x)* is any antiderivative of *f(x)* then the most general antiderivative of *f(x)* is called an <u>indefinite integral</u> and is denoted:

$$\int f(x)\ dx = F(x) + c,\quad c \text{ is any constant}$$

In this definition the is called the integral symbol, *f(x)* is called the integrand, *x* is called the integration variable and "*c*" is called the constant of integration.

HOW DO WE GO ABOUT ANTIDERIVATIVES?

1. Know it

Examples: Find $F(x)$

$2x^3 - 8x^2 + 6$ $\quad\quad$ $3sinx + 5x$ $\quad\quad$ $\frac{1}{x}$

Note: Sometimes you have to rewrite, then complete the process of antiderivative

Example	**Rewrite**	**Integrate**	**Simplify**
$\int \sqrt{x}\ dx$			
$\int \frac{3}{x^4}\ dx$			
$\int \frac{x+1}{\sqrt{x}}\ dx$			

Sometimes solve through "reverse chain rule" **Examples:**

$\int \cos(3x) - 6x\ dx$

$\int x^3 \sin(x^4)\ dx$

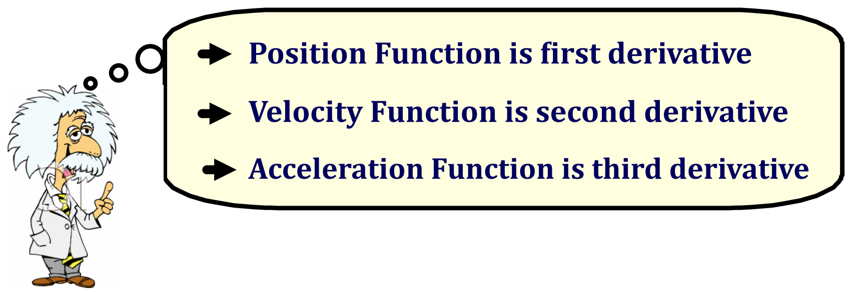
$\int \frac{2x}{x^2-5}\ dx$

But sometimes solve through:

$$Algebra\ \ Ex.\ \int (x^3+1)^2\,dx \qquad Ex.\ \int \frac{x^2-4}{x+2}\,dx \qquad Ex.\int \frac{x^3+2x+3}{x-2}\,dx$$

$$Trig.Idenities.Ex.\ \int \frac{sin2x}{cosx}\,dx$$

Position, Velocity & Acceleration Functions



Rectilinear Motion

Since $v(t) = s\,'(t)$, then position is the antiderivative of ______________________. Since $a(t) = v\,'(t)$, then velocity is the antiderivative of ______________________.

Example 1 – If the acceleration of an object is $a(t) = -2t$ m/s^2 and the initial velocity is 5 m/s and its position at 1s is 40 m, find an equation to represent the velocity and position of the object.

Example 2 – A particle is moving with $a(t) = 3\cos t - 2\sin t$, $s(0) = 0$ and $v(0) = 4$. Find the position of the particle.

2.1 - PRACTICE QUESTIONS

1. *Find the* ***GENERAL ANTIDERIVATIVE*** *of each of the following functions:*
   Verify your answers using differentiation.

$a)\quad 4x^3 - 3x^2$

$b)\quad 2x^2 - 8x + 6$

$c)\quad 3x^5 + 4x^3 - 7$

$d)\quad x^4 + \dfrac{1}{x}$

$e)\quad x^4 + \dfrac{1}{x^2} + \ln 5\ 5^x$

$f)\quad \dfrac{3}{x^5} + 2x - 7$

$g)\quad \dfrac{10}{x} + x^e$

$h)\quad x^{\frac{2}{3}} - x^3$

$i)\quad 2\cos x - 3x$

$j)\quad \cos(2x) + 4x$

$k)\quad 3\sqrt{x} - \dfrac{1}{\sqrt{x}}$

$l)\quad 10e^x - 2e^{2x}$

$m)\quad \dfrac{3}{x} - \dfrac{4}{x^2} + 2\pi$

$n)\quad \dfrac{x}{x^2+1} + 6$

$o)\quad x^2\cos(x^3)$

$p)\quad 2e^{\pi x} + x^{-2} - 5$

$q)\quad \dfrac{5x^2}{x^3 - 1}$

$r)\quad e^{\sqrt{3}\,x} - x^2 e^{x^3}$

2. If:

$a)\ \dfrac{dy}{dx} = 7e^{-2x}\ ,\ then\ y =$

$b)\ \dfrac{dy}{dx} = 3x^2 + 5x - 2\ ,\ then\ y =$

$c)\ \dfrac{dy}{dT} = \sin T + \sin 2T + \sin 3T\ ,\ then\ y =$

$d)\ \dfrac{dy}{du} = \dfrac{1}{u} - \dfrac{1}{u-1}\ ,\ then\ y =$

$e)\ \dfrac{dp}{dx} = e^x + x^e\ ,\ then\ p =$

$f)\ \dfrac{df}{dt} = 5\sec^2 3t\ ,\ then\ f =$

3. If $p = 4x^2 - 3x + 2$ find:

$a)\ the\ general\ antiderivative\ of\ p$

$b)\ y\ if\ \dfrac{dy}{dx} = p$

$c)\ y\ if\ y' = p$

$d)\ y\ if\ the\ first\ derivative\ of\ y = p$

4. *Find:*

   a) *the general antiderivative of* $x^2 + x - 8$

   b) *if* $\frac{dy}{dx} = x^2 + x - 8$ *then* $y =$

   c) *all the solutions to the differential equation* $\frac{dy}{dx} = x^2 + x - 8$

   d) *the unique solution to* $\frac{dy}{dx} = x^2 + x - 8$ *satisfying the given initial condition y(0) = 6*

   e) *the equations of all curves y = f(x) whose tangent line has a slope of* $x^2 + x - 8$

   f) *the rate of change/growth or the decay rate of change of y with respect to x is given by the expression* $x^2 + x - 8$. *Find an expression for y.*

5. *Find the position function s(t) for an object with velocity function v(t):*

   a) $v(t) = 2t^2 - 3t$ b) $v(t) = t^3 + 4t + 6$ c) $v(t) = t^2 - 5t$

6. *Find the position function s(t) for an object with velocity function v(t) and initial position s(0):*

   $a)\ v(t) = 3t - t^2,\quad s(0) = 5$ $\qquad b)\ v(t) = 6t,\quad s(0) = 7$ $\qquad v(t) = t^2 + 2t\ ,\quad s(0) = 4$

7. *Find the velocity function v(t) for an object with acceleration function a(t) and initial velocity v(0):*

   $a)\ a(t) = 5,\ v(0) = 10$ $\qquad b)\ a(t) = t - 1,\ v(0) = 1$ $\qquad c)\ a(t) = t^2 + t,\ v(0) = 0$

8. *Find the position function s(t) for an object with acceleration function a(t), initial velocity v(0) and initial position s(0).*

   $a)\ a(t) = 5,\ v(0) = 10,\ s(0) = 20$ $\qquad b)\ a(t) = t - 1,\ v(0) = 1,\ s(0) = 0$ $\qquad c)\ a(t) = t^2 + t,\ v(0) = 0,\ s(0) = 0$

9. *Find each of the following* ***INDEFINITE INTEGRALS****. Check your answers by differentiation.*

a) $\int x^2 \, dx$

b) $\int \pi \, dx$

c) $\int x^{-3} \, dx$

d) $\int 8x^3 \, dx$

e) $\int \frac{1}{x-1} \, dx$

f) $\int \frac{1}{1-x} \, dx$

g) $\int e^{4y} \, dy$

h) $\int \cos \pi y \, dy$

i) $\int e^y + y^e \, dy$

EXTENDED QUESTIONS – go to website to do.
https://mathcs.clarku.edu/~djoyce/ma120/integralpractice1.pdf

2.2 - The U-Substitution Rule

Substitution is a technique that can often be used to convert a complicated integral into a simpler one. Let's say that we start with an integral $\int f(x)dx$.

We will let u equal to some convenient x stuff --- say $u = g(x)$ which is a part of the integrand.

Steps for Integration by Substitution

1. Make a choice for u, say $u = g(x)$.
2. Compute $\frac{du}{dx} = g'(x)$.
3. Make the substitution $u = g(x)$ and $du = g'(x)dx$. (Check that no x's remain.)
4. Evaluate the resulting integral, if possible.
5. Replace u by $g(x)$ so that the final answer is in terms of x.

Examples:

$$\int \frac{1}{(2x-3)^5}\, dx$$

$$let\ u = 2x - 3$$

$$\int t^3\sqrt{3t^4 - 2}\ dt$$

Double U-Substitution

$a)\quad \int x(x+1)^5\,dx$

$b)\quad \int x^3(x^2+1)^{\frac{3}{2}}\,dx$

2.2 - PRACTICE QUESTIONS

1. *Find the antiderivative by* ***U-Substitution***.

$a)\ \int (3x-5)^{17}\,dx$

$u = 3x-5$

$b)\ \int \frac{1}{(4x+7)^6}\,dx$

$u = 4x+7$

$c)\ \int x\sqrt{x^2+9}\,dx$

$u = x^2+9$

$d)\ \int x^2\sqrt{2x^3-4}\,dx$

$u = 2x^3-4$

$e)\ \int \frac{(\ln x)^{10}}{x}\, dx$

$f)\ \int \frac{5x}{5+2x^2}\, dx$

$g)\ \int \frac{4x}{\sqrt{x^2+1}}\, dx$

$h)\ \int e^{t^2+\ln(t)}\, dt$

EXTENDED QUESTIONS

Integration by "Double Substitution" - Evaluate each indefinite integral.

1. $\int x\sqrt{1+x}\ dx$

2. $\int \frac{x}{\sqrt{x+1}}\,dx$

3. $\int \frac{x^3}{x^2-3}\,dx$

4. $\int \frac{13x^7}{\sqrt{3x^4-5}}\,dx$

2.3 - Integration by Parts

Formula: $\int u\,dv = uv - \int v\,du$

The key thing in integration by parts is to choose u and dv correctly.

The acronym **ILATE** is good for picking u.

ILATE stands for >

I	Inverse Trigonometric Functions $\arcsin x, \arctan x, \ldots$
L	Logarithmic Functions $\ln x, \log x, \log_2 x, \ldots$
A	Algebraic Functions $x, x^2, x^3, 2x^5, \ldots$
T	Trigonometric Functions $\sin x, \cos x, \tan x, \ldots$
E	Exponential Functions $e^x, e^{2x}, 2^x, 3^{-x}, \ldots$

Formula: $\int u\,dv = uv - \int v\,du$

Examples:

$\int xe^x\ dx$

$\int x\cos(2x)\ dx$

$\int \frac{\ln x}{x^2}\,dx$

Formula: $\int u\,dv = uv - \int v\,du$

You Try:

$$\int \mathrm{x}e^{2x}\ dx$$

$$\int xsin(3x-2)\ dx$$

$$\int lnx\ dx$$

Note: Sometimes you may have to integrate by Parts more than once.

Example:

$$\int (lnx)^2 \; dx$$

You Try:

$$\int e^{7x} cos(2x) \; dx$$

2.3 - PRACTICE QUESTIONS

1. *Evaluate each indefinite integral using* ***Integration by Parts****.*

$a)\ \int x \sin(x)\, dx$

$b)\ \int x \cos(4x)\, dx$

$c)\ \int 2x^2 e^x\ dx$

$d)\ \int x^2\, ln|x|\, dx$

EXTENDED QUESTIONS

Evaluate each indefinite integral using integration by parts. *u* and *dv* are provided.

1) $\int xe^x \, dx;\ u = x,\ dv = e^x \, dx$

2) $\int x\cos x \, dx;\ u = x,\ dv = \cos x \, dx$

3) $\int x \cdot 2^x \, dx;\ u = x,\ dv = 2^x \, dx$

4) $\int \sqrt{x} \ln x \, dx;\ u = \ln x,\ dv = \sqrt{x} \, dx$

Evaluate each indefinite integral.

5) $\int xe^{-x} \, dx$

6) $\int x^2 \cos 3x \, dx$

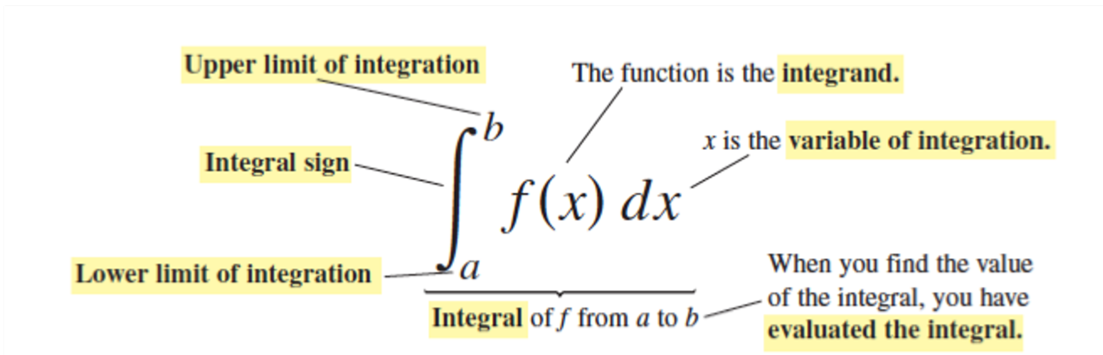
7) $\int \frac{x^2}{e^{2x}} \, dx$

8) $\int x^2 e^{5x} \, dx$

9) $\int \ln (x + 3) \, dx$

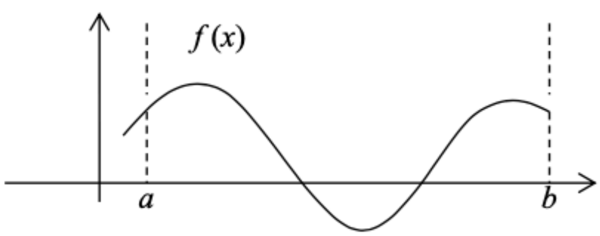
10) $\int \cos 2x \cdot e^{-x} \, dx$

2.4 – Definite Integrals



A definite integral can be interpreted as a _______ __________, that is, a difference of areas.

Eg. Suppose we have the following function:



Any area above the x-axis and below the curve is counted as ________________.

Any area below the x-axis and above the curve is counted as ________________.

$$\therefore \int_a^b f(x)dx =$$

Note: This is known as "**FIRST FUNDAMENTAL THEOREM OF CALCULUS**"

Find the Definite Integral:

Examples:

$a)\ \int_{-1}^{3} x^2 + 2\ dx$

$b)\ \int_{e}^{4} \frac{\ln t}{t}\ dt$

$c)\ \int_{a}^{b} 4t^3\ \ dt$

$d)\ \int_{2}^{5} 2b\ dx$

Ex: Solve for x:

$$\int_{x}^{2} x^3\ dx = 0$$

You Try:

$Ex.\ 1)\ \int_{1}^{4} \left(2x^2 + 4x + 1\right) dx$

$Ex.\ 2)\ \int_{0}^{4} \frac{u^2 - 3}{\sqrt{u}} du$

2.4 - PRACTICE QUESTIONS

1. *Find the **EXACT VALUE** of each of the following DEFINITE INTERGRALS.*

$a)\ \int_1^3 2x\ dx$ $\qquad b)\ \int_{-1}^1 x^2\ dx$ $\qquad c)\ \int_{-2}^{-1} x-6\ dx$

$d)\ \int_1^{\sqrt{2}} x\ dx$ $\qquad e)\ \int_\pi^{2\pi} e^x\ dx$ $\qquad f)\ \int_{-2}^{-1} x^2+6\ dx$

$g)\ \int_1^4 3y\ dy$ $\qquad h)\ \int_1^4 3y\ dm$ $\qquad i)\ \int_e^4 \frac{\ln t}{t}\ dt$

$j)\ \int_0^{\frac{\pi}{2}} \cos t\ dt$ $\qquad k)\ \int_2^5 e^x\ dr$ $\qquad l)\ \int_{-3}^{-1} xe^{x^2}\ dm$

2. *Simplify:*

$a)\ \int_a^b 2x\ dx$ $\qquad b)\ \int_a^b 3x^2\ dx$ $\qquad c)\ \int_0^b x^2\ dx$

$d)\ \int_a^b 2x\ dm$ $\qquad e)\ \int_0^b e^p\ dp$ $\qquad f)\ \int_\pi^4 \frac{2}{t}\ dt$

3. *Simplify:*

$a) \int_a^5 2x \ dx$

$b) \int_0^b x^3 \ dx$

$c) \int_a^0 e^x \ dx$

$d) \int_{4m}^{4m} \frac{y}{y^3+1} \ dy$

$e) \int_{2p}^{4p} y \ dt$

$f) \int_{-4a}^{6a} c \ dm$

4. *Solve for x:*

$a) \int_1^x 2m \ dm = 8$

$b) \int_x^1 4 \ dy = 16$

$c) \int_x^5 2t+3 \ dt = 40$

$d) \int_2^x 4k \ dy = 8x$

$e) \int_1^4 x \ dq = 15$

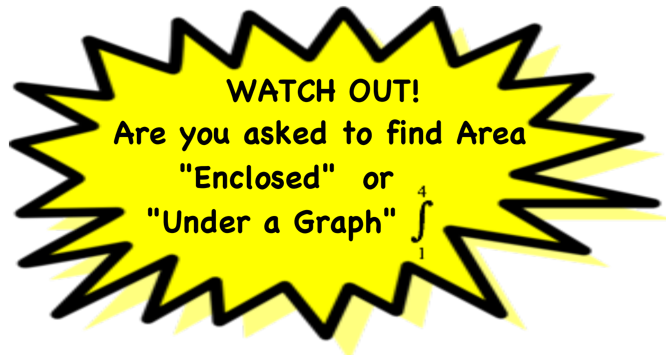
$f) \int_2^x r \ dr = 1$

2.5 – Area Under a Curve Bounded to x-axis

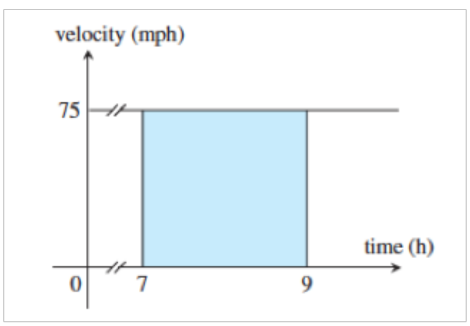
All this integration leads us to the key concept of finding "Area under a Curve" or "Area Bounded between two Curves".

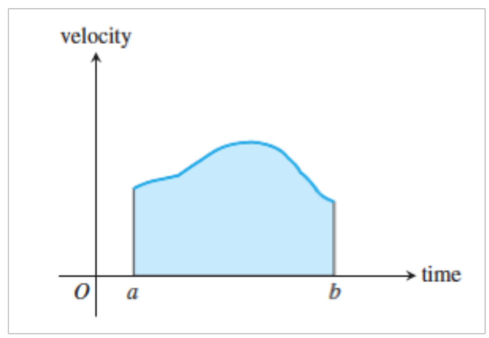
We will explore 3 types of Area Enclosed or Bounded

1. **Between a curve and x-axis**
2. **Between a curve and y-axis**
3. **Between two curves**



The concept of Area



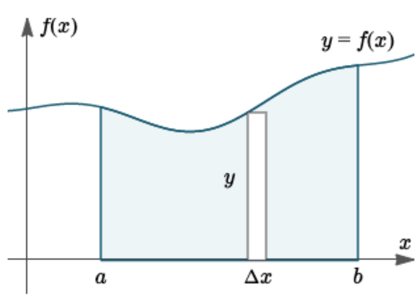


Distance travelled and the area in this case are both found by multiplying the rate by the change in time.

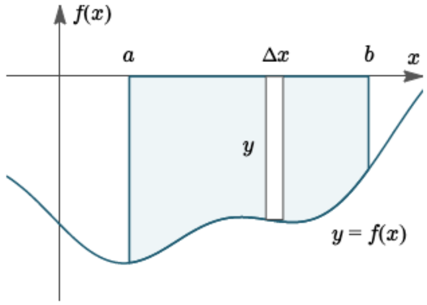
1. Area Under a Curve by Integration Bounded to x-axis

Case 1: **Curves which are entirely above the x-axis.**

$$\text{Area} = \int_a^b f(x)\,dx$$



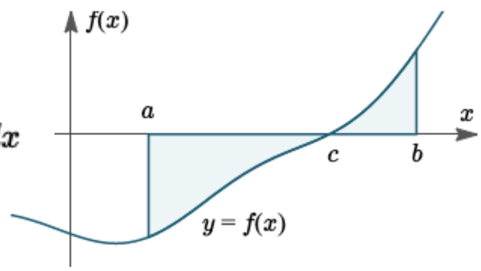
Case 2: **Curves which are entirely below the x-axis**



$$\text{Area} = \left| \int_a^b f(x)\,dx \right|$$

Case 3: **Part of the curve is below the x-axis, part of it is above the x-axis**

$$\text{Area} = \left| \int_a^c f(x)\,dx \right| + \int_c^b f(x)\,dx$$

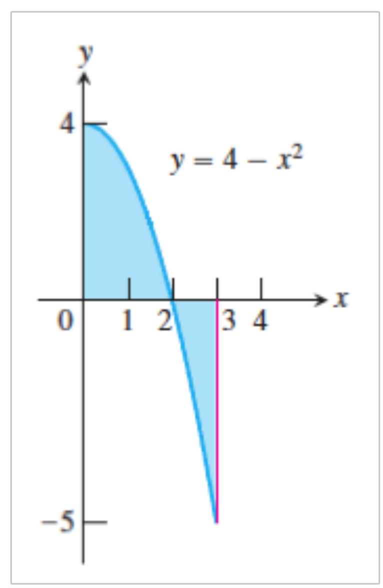


NOTE: It is important to sketch the situation before you start.

WATCH OUT!
Are you asked to find Area "Enclosed" or "Under a Graph" $\int_1^4$

Let's look at the difference

EXAMPLE:

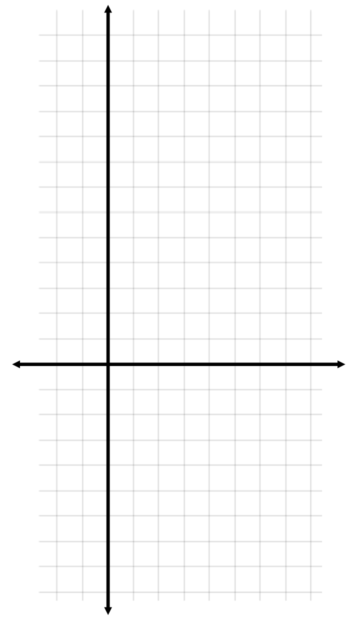


a) $\int_0^3 4 - x^2$

b) Find the area of $y = 4 - x^2$ enclosed with

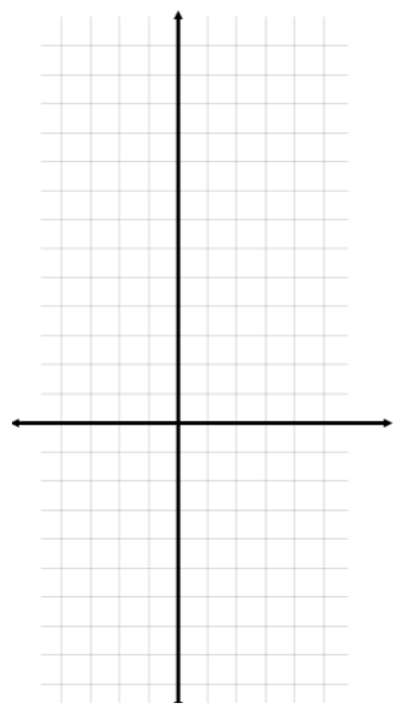
Example 1: Find the Exact Area between the Given Curve & the *x*-axis over the Given Interval.

$y = 3x \qquad -1 \le x \le 3$



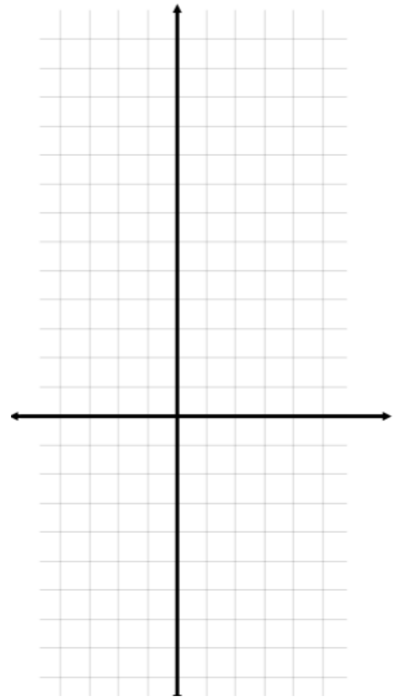
Example 2: Find the Exact Area between the Given Curve & the *x*-axis over the Given Interval.

$y = x^3 + 2x^2 - 3x \qquad -3 \le x \le 1$



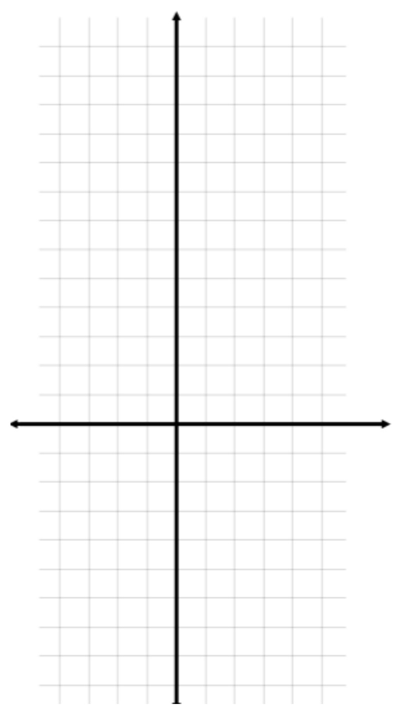
You Try 1: Find the Exact Area between the Given Curve & the x-axis over the Given Interval.

$y = x^3 \quad -2 \le x \le 2$



You Try 2: Find the Exact Area between the Given Curve & the x-axis over the Given Interval.

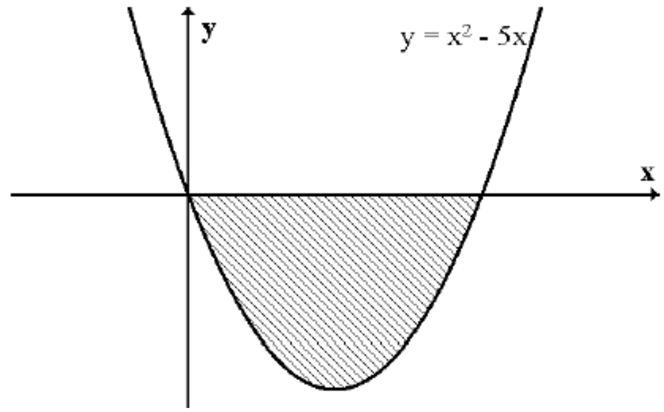
$y = x^2 - 1 \quad 0 \le x \le 2$



2.5 - PRACTICE QUESTIONS

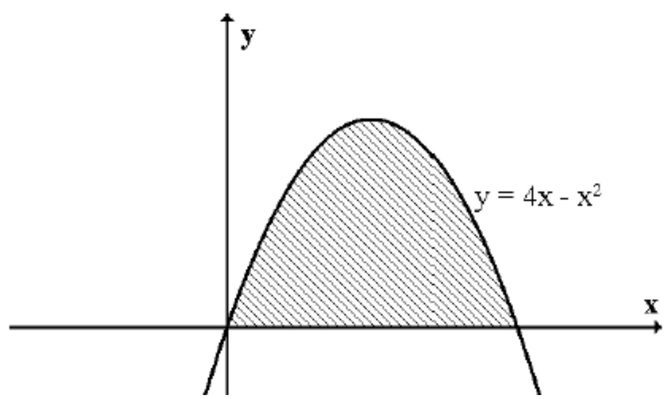
1. The diagram opposite shows the graph of $y = x^2 - 5x$.

   Calculate the shaded area.



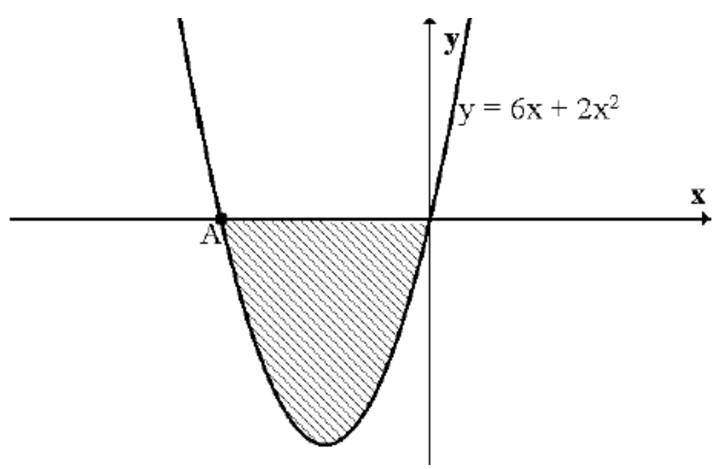
2. The diagram shows the graph of $y = 4x - x^2$.

   Calculate the area between the curve and the x-axis.



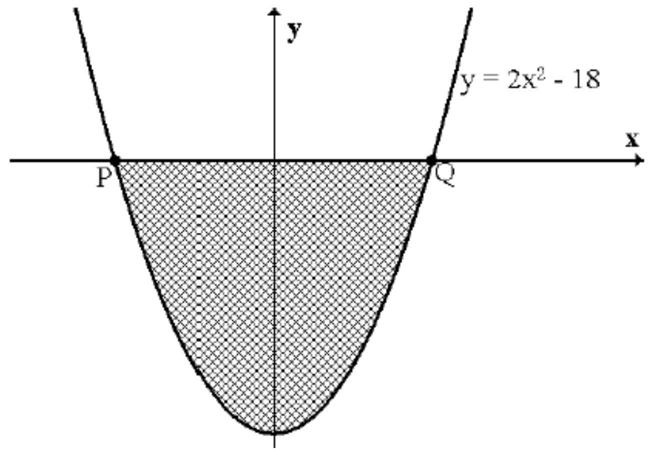
3. The diagram shows part of the graph of $y = 6x + 2x^2$.

   (a) Find the coordinates of A.
   (b) Calculate the shaded area.



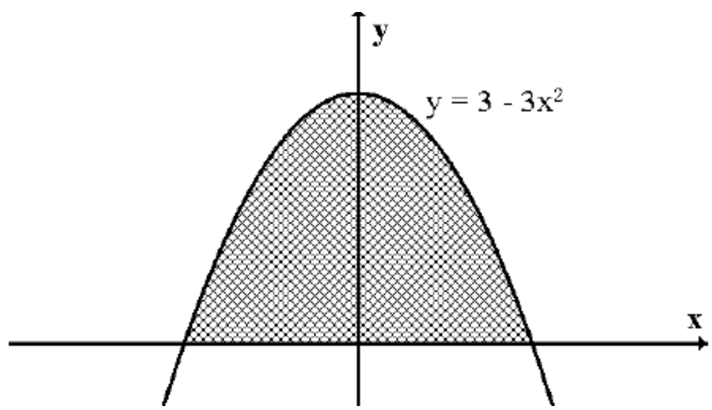
4. The dagram shows part of the graph of
$y = 2x^2 - 18$.

(a) Calculate the coordinates of P and Q.
(b) Find the shaded area.



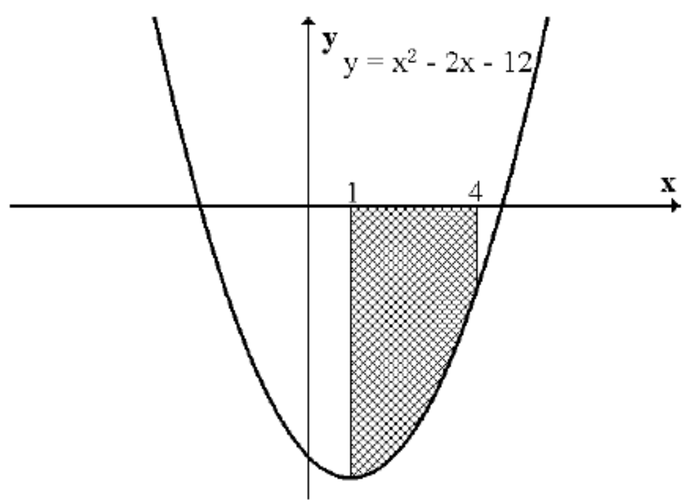
5. The diagram shows part of the graph of
$y = 3 - 3x^2$.

Calculate the shaded area.



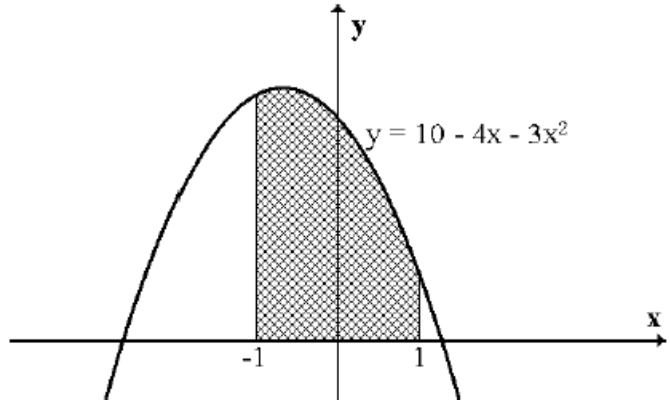
6. The diagram shows the graph of
$y = x^2 - 2x - 12$.

Calculate the shaded area.



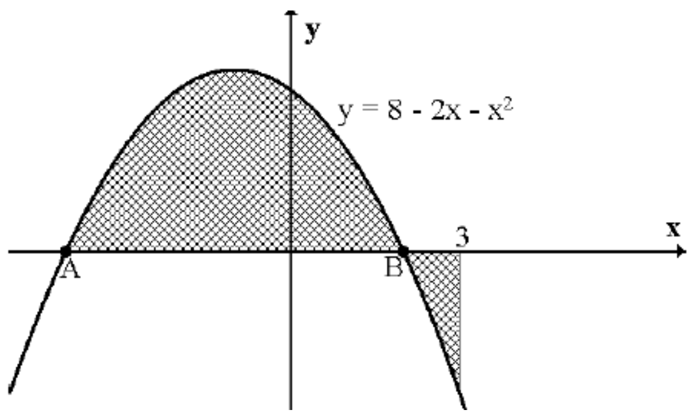
7. The diagram shows part of the graph of $y = 10 - 4x - 3x^2$

Calculate the shaded area.



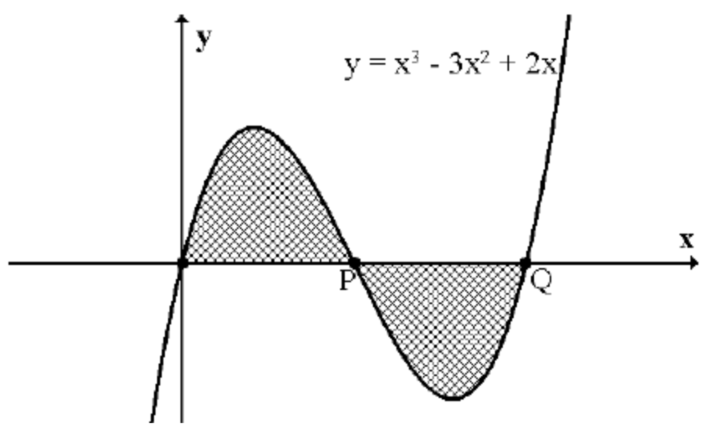
8. The diagram shows the graph of $y = 8 - 2x - x^2$.

(a) Find the coordinates of A and B.
(b) Calculate the shaded area.



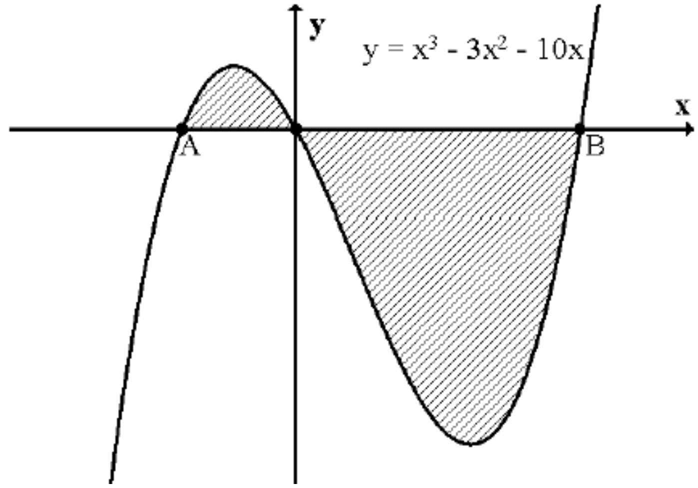
9. The diagram opposite shows part of the graph of $y = x^3 - 3x^2 + 2x$.

(a) Find the coordinates of P and Q.
(b) Calculate the shaded area.



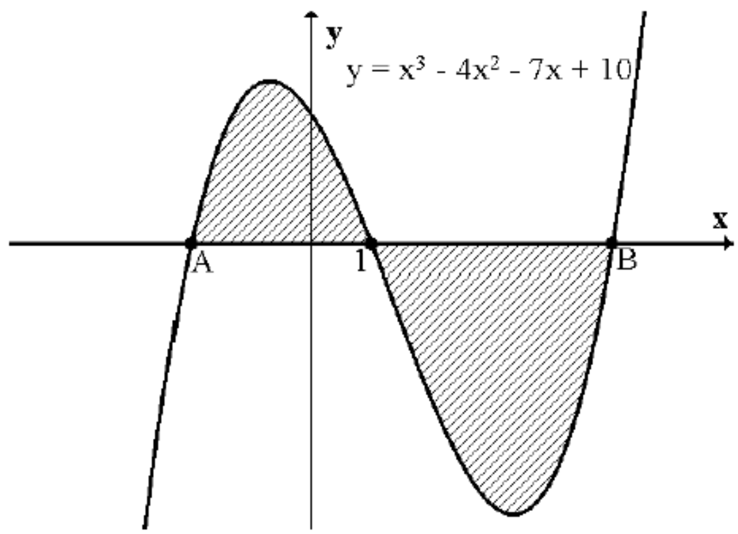
10. The diagram shows the graph of
$y = x^3 - 3x^2 - 10x$.

(a) Find the coordinates of A and B.
(b) Calculate the shaded area.



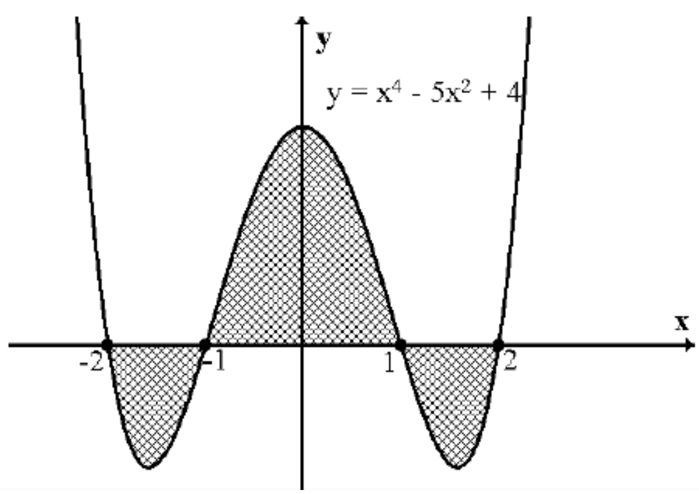
11. The diagram shows the graph of
$y = x^3 - 4x^2 - 7x + 10$.

(a) Find the coordinates of A and B.
(b) Calculate the shaded area.

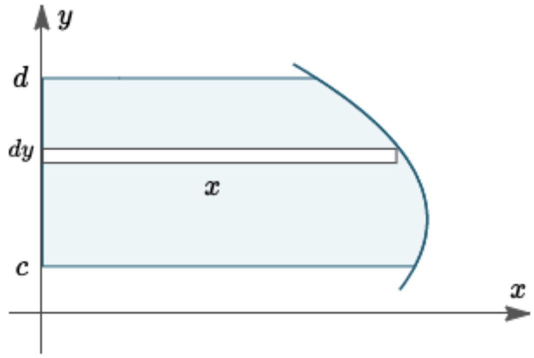


12. The diagram shows the graph of
$y = x^4 - 5x^2 + 4$.

Calculate the shaded area.



2.6 – Area Under a Curve and Bounded to *y*-axis



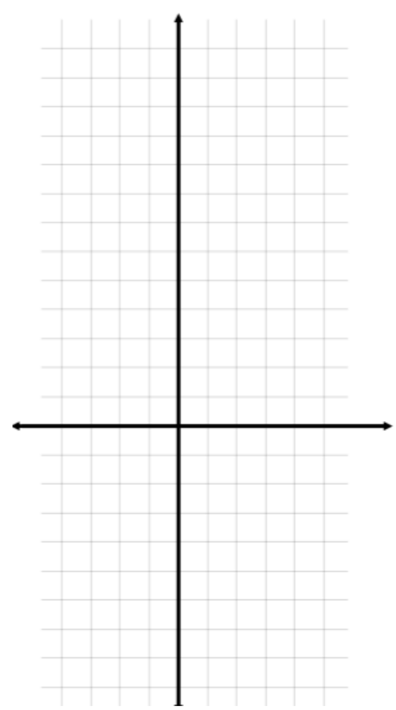
$$A = \int_c^d f(y)\,dy$$

There are 2 Methods

Method #1 – "Box Method" (*subtraction)*

Example: Find the Exact Area between the Given Curve & the y-axis over the Given Interval

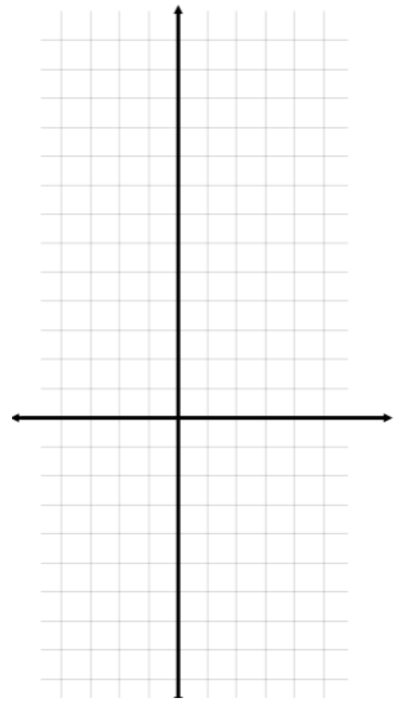
$y = x^2, \qquad 0 \le x \le 3$



Method #2 – Integrate with respect to *y*-axis

NOTE: *Watch out as in area to the x-axis we had to sperate the area above and below x-axis, same goes for area to the y-axis…separate right and left.*

Example: Find the area bounded by the curve $y = x^3$ the *y*-axis and the lines $y = -1$ and $y = 8$



You Try: Find the area bounded by the curve $y = \sqrt{x+1}$ the *y*-axis and the domain $-1 \leq x \leq 3$.

2.6 - PRACTICE QUESTIONS

1. *Find the* ***Exact Area*** *between the given curve and the y-axis over the given interval.*

$a)\ y = 3x, \quad 0 \le x \le 4$

$b)\ y = x^2, \quad 0 \le x \le 3$

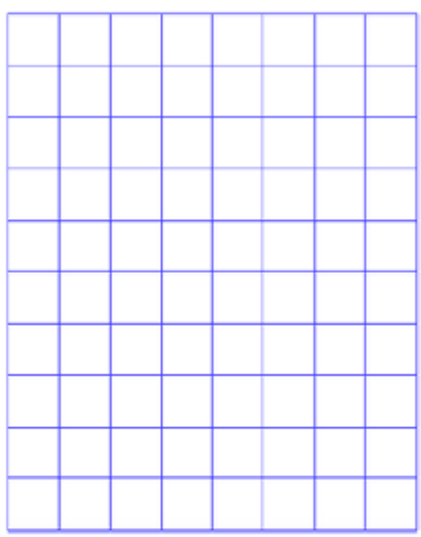
$c)\ y = 4x^3, \quad 0 \le x \le 3$

$d)\ y = \cos x, \quad 0 \le x \le \frac{\pi}{2}$

$e)\ y = e^x, \quad 0 \le x \le 2$

$f)\ y = -4x, \quad 0 \le x \le 4$

2. *Find the area of the region bounded by* $y = -2 + x$, $x = 0$ *and* $y = 3$.

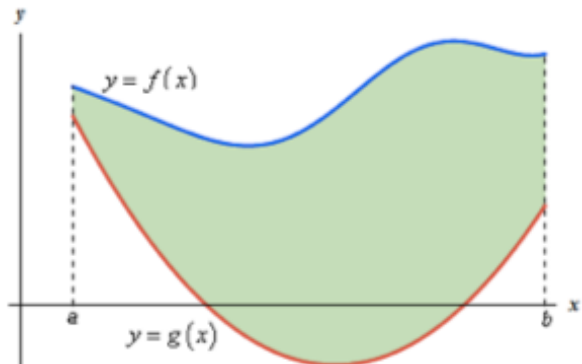


2.7 – Area Enclosed Between Two Curves with respect to x

In this section we are going to look at finding the area between two curves. There are actually two cases we are going to be looking at.

Case 1

In the first case we want to determine the area between $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$. We are also going to assume that $f(x) \geq g(x)$. Take a look at the following sketch to get an idea of what we're initially going to look at.



$$\boxed{A = \int_a^b f(x) - g(x)\,dx}$$

$$\boxed{A = \int_a^b \begin{pmatrix} \text{upper} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{lower} \\ \text{function} \end{pmatrix} dx, \qquad a \leq x \leq b}$$

Note: Not always $y = f(x)$ is greater than $y = g(x)$.

STEPS – to solving these area problems

1. You must determine which function is upper and which is lower functions.

   Two way to determine this:
   1. Draw a sketch, or
   2. Pick a number between the interval of the x-axis and plug into each function

2. Upper function minus the Lower function

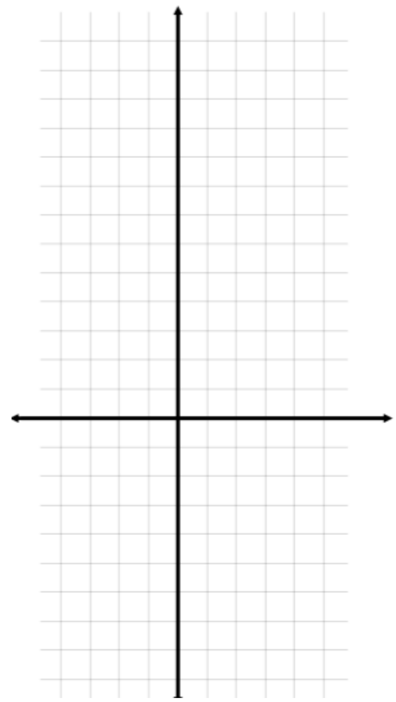
3. Simplify before you integrate.

4. Integrate using the upper and lower limit of integration

Note: you do not have to worry about parts of the function below x-axis.

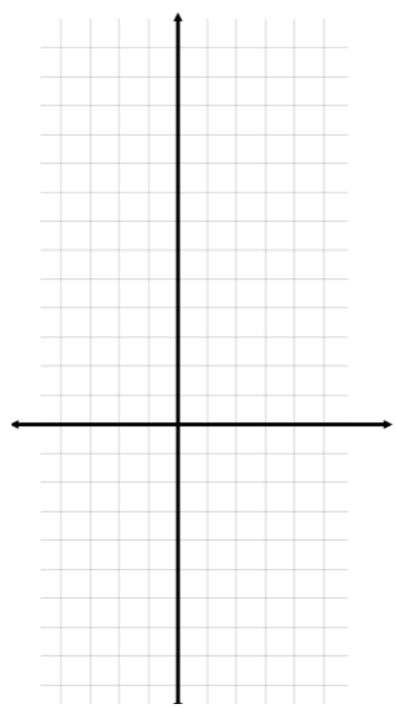
Example: Find the Exact Area between the two curves over the given interval

$f(x) = -2x + 1; \quad g(x) = -4x - 5, \qquad 0 \le x \le 2$

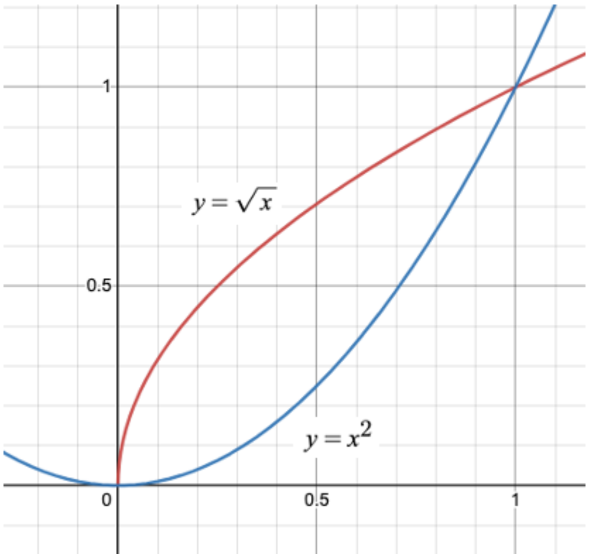


You Try: Find the Exact Area between the two curves over the given interval

$f(x) = x^2 + 1; \quad g(x) = x, \qquad [-1, 2]$

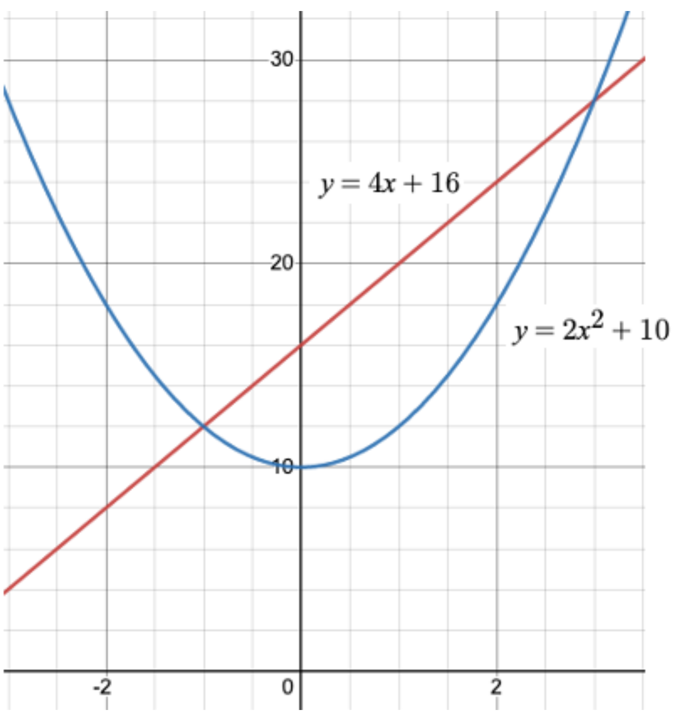


Example: Find the Exact Area Enclosed by the two curves.



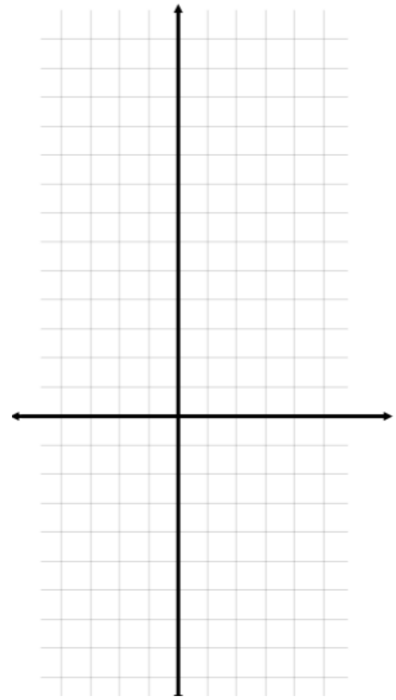
What do you notice that is different from the previous examples?

You Try: Find the Exact Area Enclosed by the two curves.



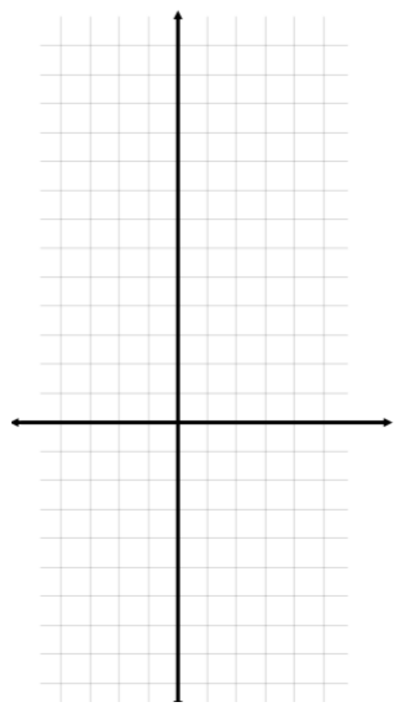
Example: Find the Exact Area Enclosed by the two curves.

$f(x) = 4x; \quad g(x) = x^2$



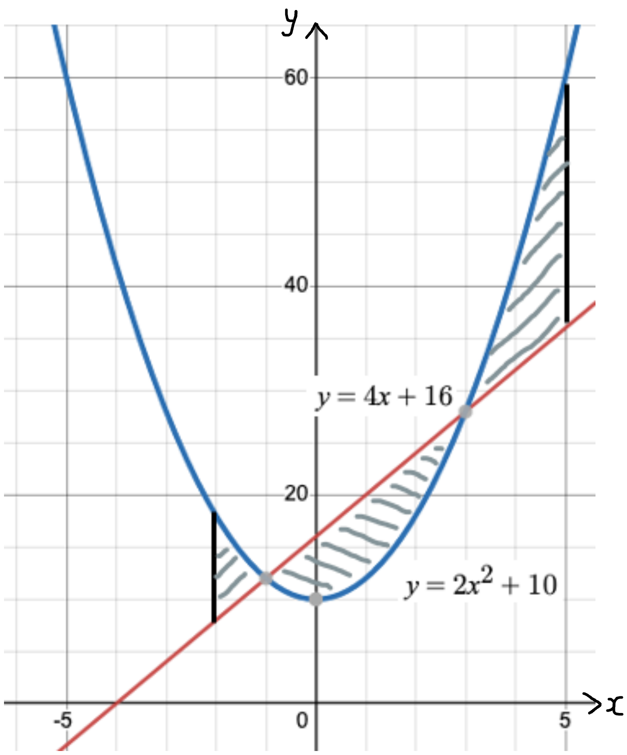
You Try: Find the Exact Area Enclosed by the two Curves

$f(x) = x^2 \;; \qquad g(x) = 2x - x^2$

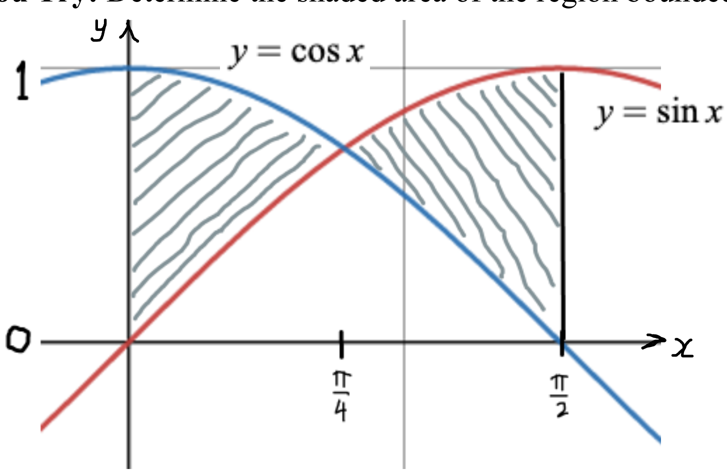


*****Watch out for when the Upper & Lower Functions Switch***

Example: Determine the shaded area of the region bounded by
$y = 2x^2 + 10, \;\; y = 4x + 16, \;\; x = -2 \; and \; x = 5$



You Try: Determine the shaded area of the region bounded by $y = sinx, \;\; y = cosx, \;\; [0, \frac{\pi}{2}]$



2.7 - PRACTICE QUESTIONS

Determine the shaded area the region enclosed for each:

1. *Find the **exact area** between the two curves f(x) and g(x) over the given interval.*

$a)\ \ f(x) = 2,\ \ g(x) = -3;\quad -1 \leq x \leq 4$

$b)\ \ f(x) = -2x + 1,\ \ g(x) = -4x - 5\ ;\quad 0 \leq x \leq 2$

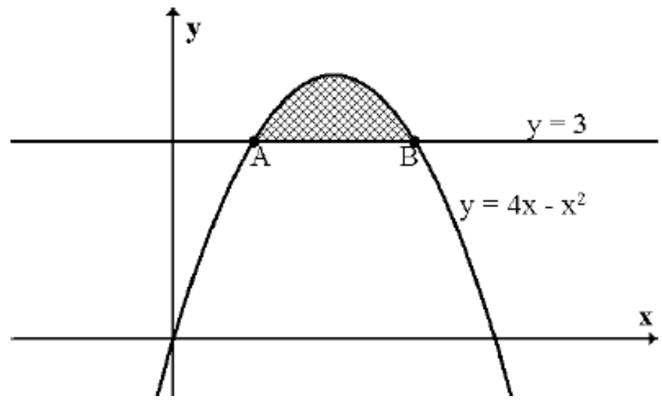
$a)\ \ f(x) = e^x,\ \ g(x) = \frac{1}{2}x\ ;\quad -2 \leq x \leq 0$

$b)\ \ f(x) = sinx,\qquad g(x) = -2x\ ;\qquad 0 \leq x \leq \pi$

2.

The diagram opposite shows the curve $y = 4x - x^2$ and the line $y = 3$.

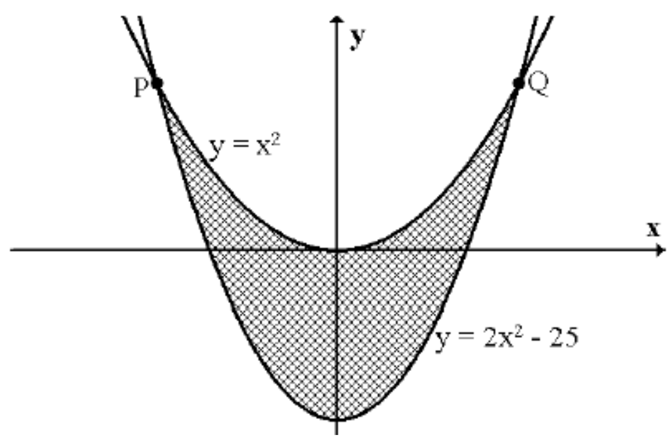
(a) Find the coordinates of A and B.
(b) Calculate the shaded area.



3.

The curves with equations $y = x^2$ and $y = 2x^2 - 25$ intersect at P and Q.

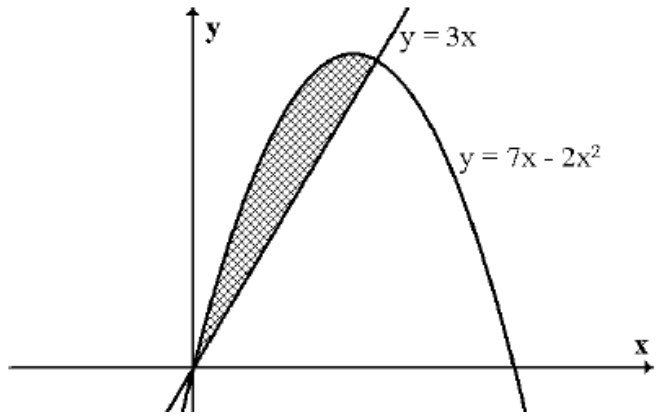
Calculate the area enclosed between the curves.



4.

The diagram opposite shows the curve $y = 7x - 2x^2$ and the line $y = 3x$.

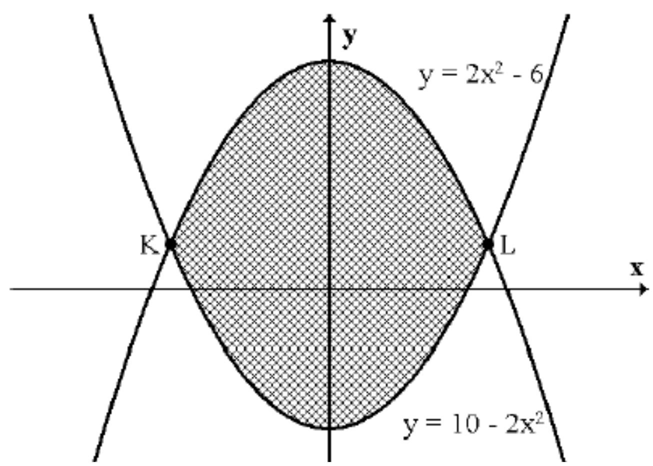
Calculate the shaded area.



5.

The curves with equations $y = 2x^2 - 6$ and $y = 10 - 2x^2$ intersect at K and L.

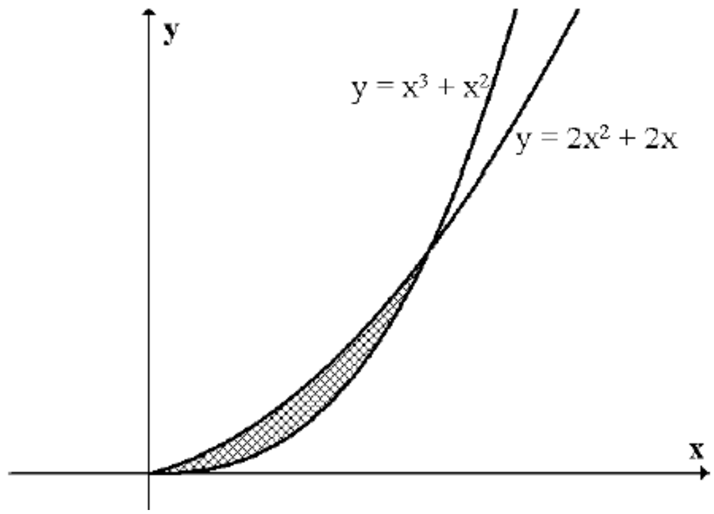
Calculate the area enclosed by these two curves.



6.

The diagram opposite shows part of the curves $y = x^3 + x^2$ and $y = 2x^2 + 2x$.

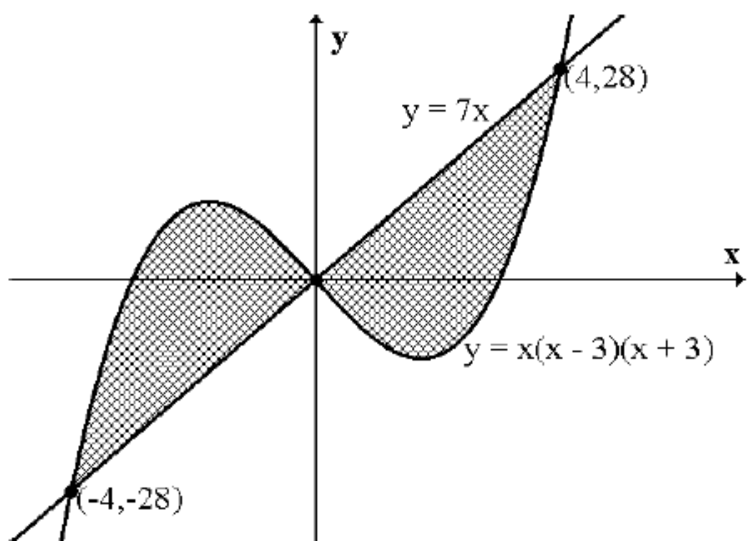
Calculate the shaded area.



7.

. The curve $y = x(x - 3)(x + 3)$ and the line $y = 7x$ intersect at the points $(0,0)$, $(-4,-28)$ and $(4,28)$.

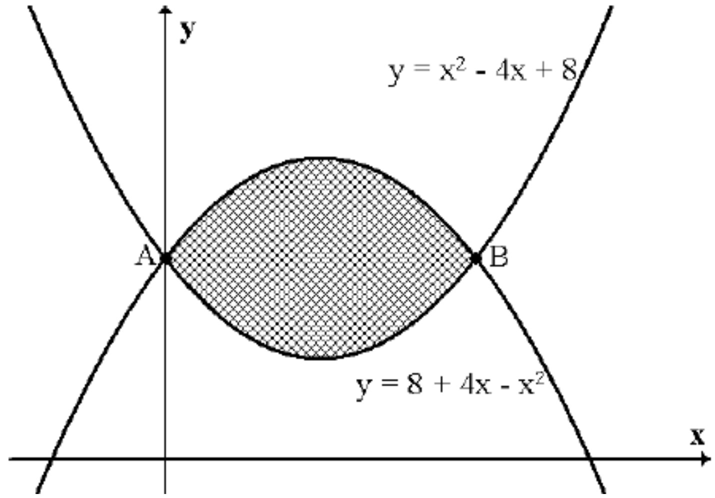
Calculate the area enclosed by the curve and the line.



8.

The parabolas $y = x^2 - 4x + 8$ and $y = 8 + 4x - x^2$ intersect at A and B.

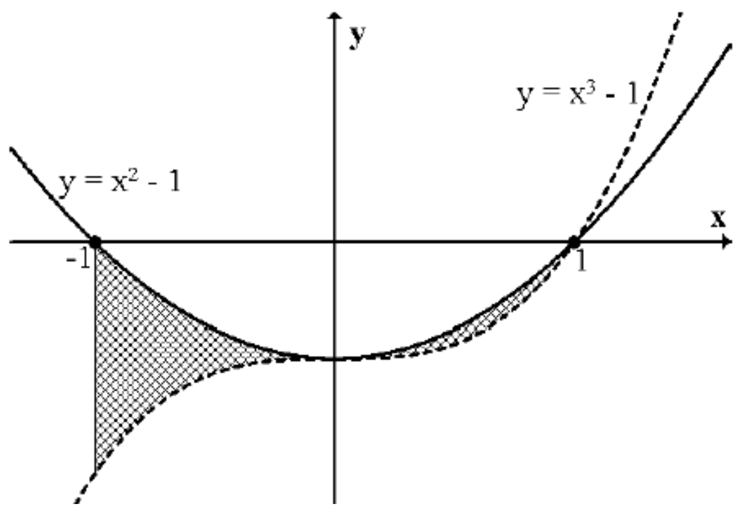
(a) Find the coordinates of A and B.
(b) Calculate the shaded area.



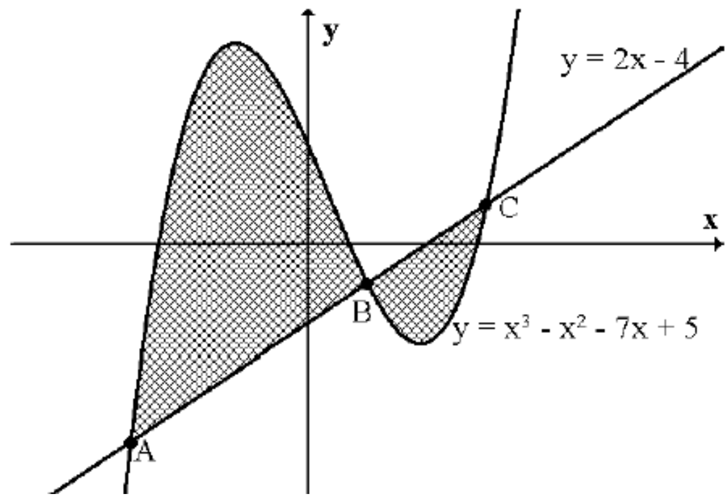
9.

The diagram shows parts of the curves $y = x^3 - 1$ and $y = x^2 - 1$.

Calculate the shaded area.



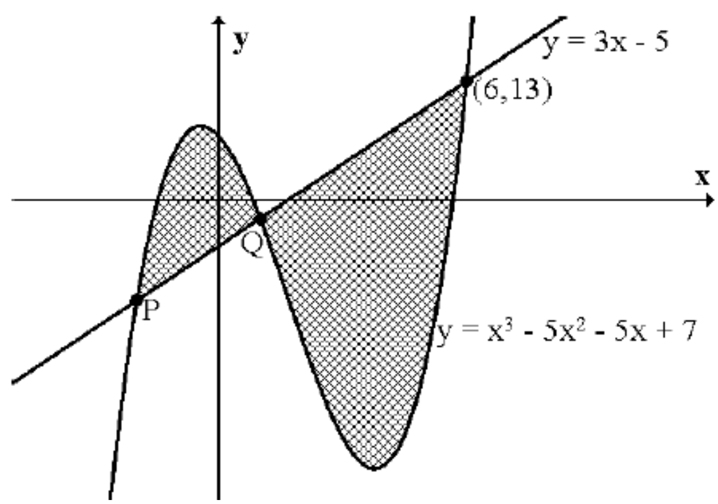
10.



The curve $y = x^3 - x^2 - 7x + 5$ and the line $y = 2x - 4$ are shown opposite.

(a) B has coordinates (1,-2). Find the coordinates of A and C.
(b) Hence calculate the shaded area.

11.



. The diagram shows the line $y = 3x - 5$ and the curve $y = x^3 - 5x^2 - 5x + 7$.

(a) Find the coordinates of P and Q.
(b) Calculate the shaded area.

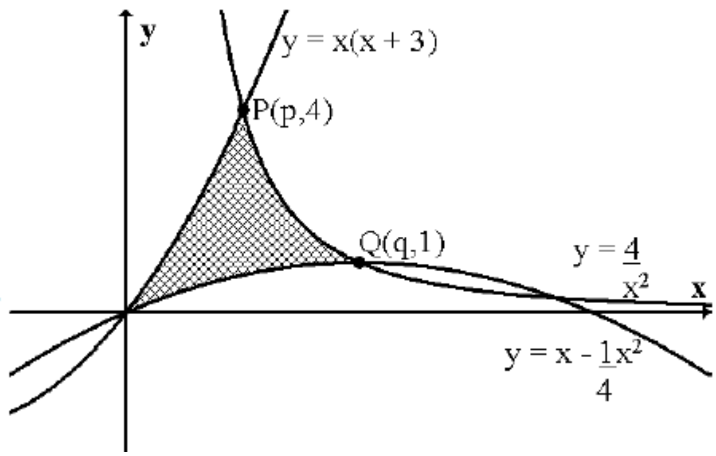
EXTENDED QUESTION

The diagram opposite shows an area enclosed by 3 curves:

$y = x(x + 3)$, $y = \dfrac{4}{x^2}$ and $y = x - \dfrac{1}{4}x^2$

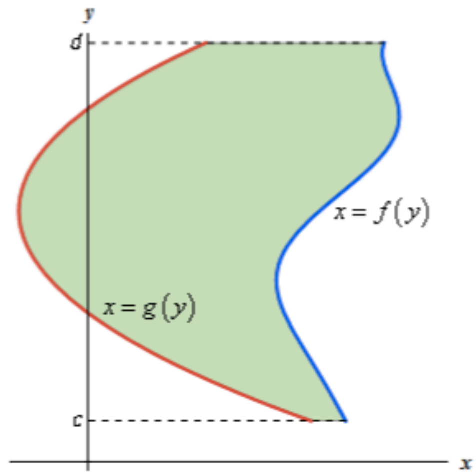
(a) P and Q have coordinates (p,4) and (q,1). Find the values of p and q.

(b) Calculate the shaded area.



2.8 – Area Enclosed Between Two Curves with respect to *y*

The second case is almost identical to the first case. Here we are going to determine the area between $x = f(y)$ and $x = g(y)$ on the interval $[c, d]$ with $f(y) \geq g(y)$.



$$\boxed{A = \int_c^d f(y) - g(y)\,dy}$$

$$\boxed{A = \int_c^d \begin{pmatrix} \text{right} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{left} \\ \text{function} \end{pmatrix} dy, \qquad c \leq y \leq d}$$

Note: Not always $x = f(y)$ is greater than $x = g(y)$.

STEPS – to solving these area problems

1. You must determine which function is right and which is left functions.

    Two way to determine this:
    1. Draw a sketch, or
    2. Pick a number between the interval of the *y*-axis and plug into each function

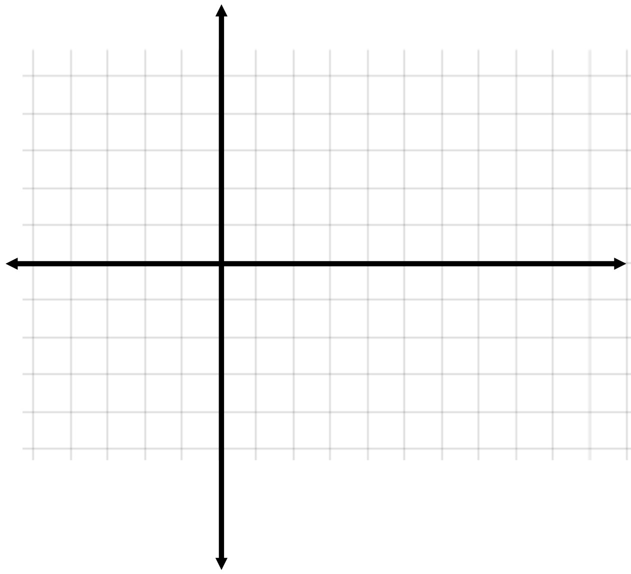
2. Right function minus the Left function

3. Simplify before you integrate.

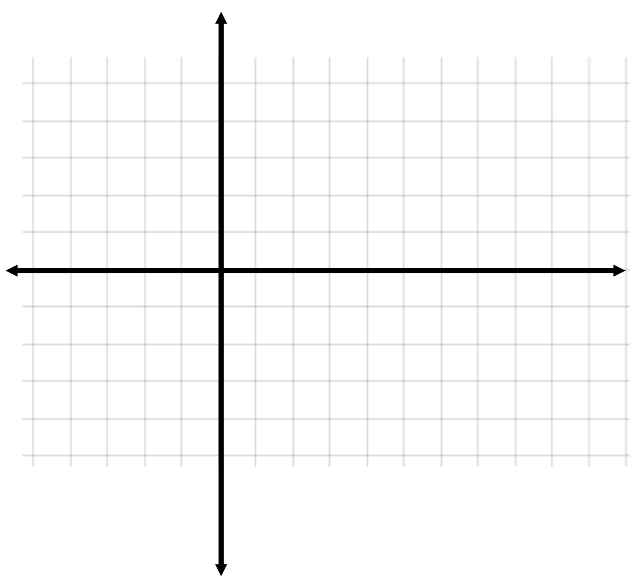
4. Integrate using the upper and lower limit of integration

Note: you do not have to worry about parts of the function below *x*-axis.

Example: Find the area of the region enclosed by the following curves:
$x = y^2 - 2$, and $x = y$.



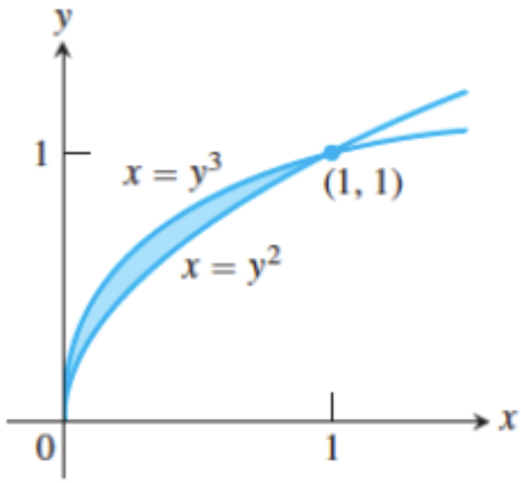
You Try: Find the exact area enclosed by the two curves:
$x = y^3 - y$, and $x = 1 - y^4$.



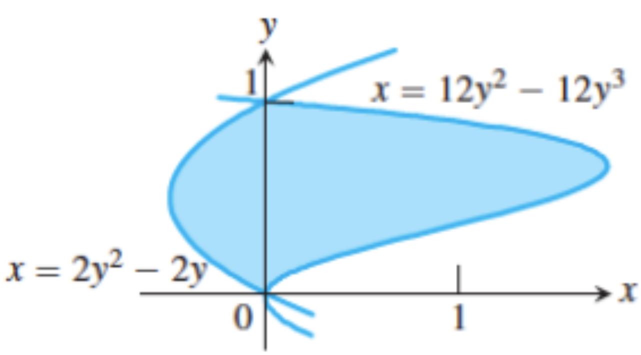
2.8 - PRACTICE QUESTIONS

Find the area of the shaded region between two curves – y-axis

1)



2)

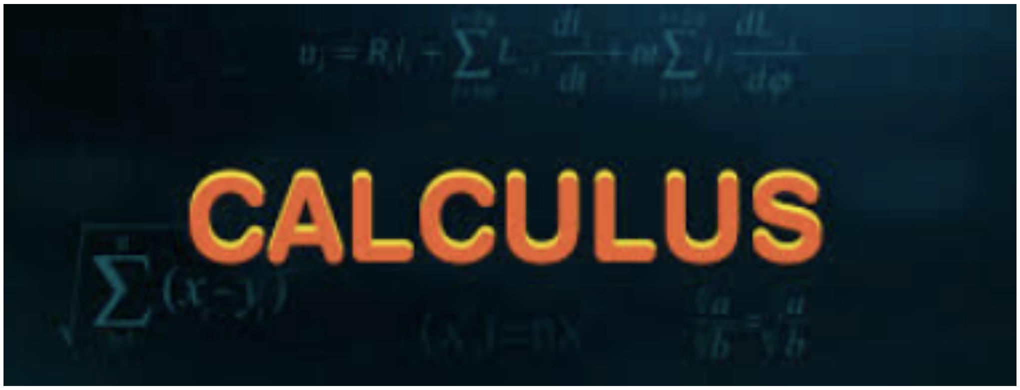


3. Find the area of the region(s) enclosed by the graphs of $x - y^2 = 0$ and $x + 2y^2 = 3$.

4. Find the area of the region enclosed by the following curves: $x = y^2 - 2$, $and \;\; x = y$.

5. Find the area of the region enclosed by the following curves: $x = \frac{1}{2}y^2 - 3$, and $y = x - 1$.

6. Find the area of the region enclosed by the following curves: $x = y^3 - y$, and $x = 1 - y^4$.



SEMINAR NOTES

UNIT 3

3.1 - The Limit of a Function

Suppose I have one whole pizza and I decide to eat half of it. (1⁄2) (mmm) I can still hear my stomach growling from hunger, so I decide to eat half of what's remaining. (___). I am starting to get annoyed, because I am STILL hungry. So I decide to eat half of what I just ate. (___). And so it goes on... Notice that I will never eat the whole pizza under this premise, but the more I eat, the closer I will get to having eaten the whole pizza. We say that the *limit* is 1.

1. Limits

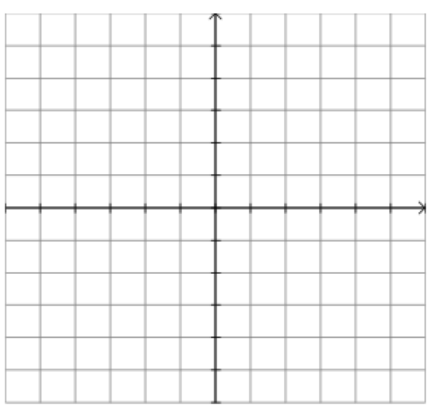
Limits can be used to describe how a function behaves as the independent variable x (input number) approaches a certain value, y (output number), even if the output value does not exist.

Example 1: Consider the function $f(x) = 2x - 3$ and suppose we select input numbers, x, closer and closer to the number 4. Fill in the input-output table and take note of what happens to the output numbers.

Numerical representation:

x	2	3.6	3.9	3.99	3.999	4.001	4.01	4.1	4.8	5
$f(x)$										

Here is a graphical representation:



Verbally:
As x approaches 4, ___________________

As $x \to 4$, ___________

OR algebraically: $\lim\limits_{x \to}$

*We could have evaluated at x = 4 (this does not always work!)

Definition of a Limit

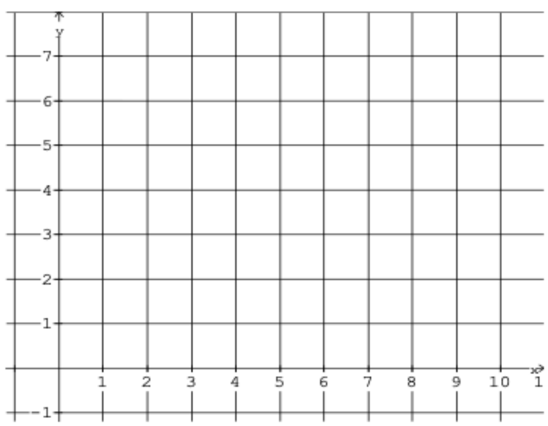
Note: The concept of a limit is CENTRAL to calculus – we will concentrate today on an intuitive introduction to limits

Limits can be used to describe how a function behaves as the independent variable moves towards a certain value.

Definition: $\lim_{x \to a} f(x) = L$ **is read, "the limit of f of x as x approaches a equals L". It means that as**

Example 2: Make a conjecture about the value of the limit $\lim_{x \to 9} \frac{x-9}{\sqrt{x}-3}$ numerically and graphically.

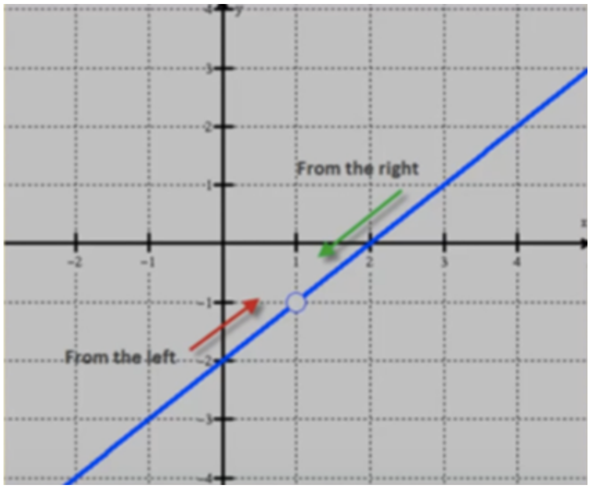
x	8	8.9	8.99	8.999	9	9.001	9.01	9.10	10
$f(x)$									



*** Note: It is important to graph as well as use a table of values so that you can visually see what is happening.**

The existence of a limit as x → a doesn't always depend on how the function may or may not be defined at a.

Example: Let's consider $\lim_{x \to 1} f(x) =$ t



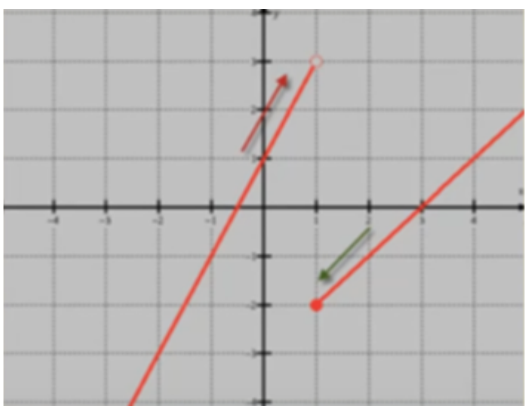
From the left

From the right

x	f(x)
0.1	-1.9
0.5	-1.5
0.9	-1.1
0.99	-1.01
0.999	-1.001
1	Und
1.001	-0.999
1.01	-0.99
1.1	-0.9
1.5	-0.5
1.9	-0.1

Let's consider the following limits:

$\lim_{x \to 1^+} f(x) =$ $\lim_{x \to 1^-} f(x) =$ $\lim_{x \to 1} f(x) =$

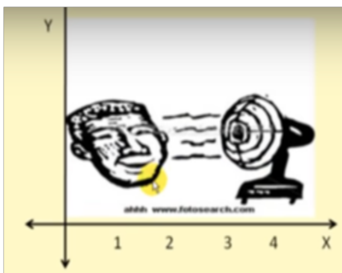


From the right

x $(x \to 1^+)$	f(x)
1.2	-1.8
1.1	-1.9
1.01	-1.99
1.001	-1.999
1	?

From the left

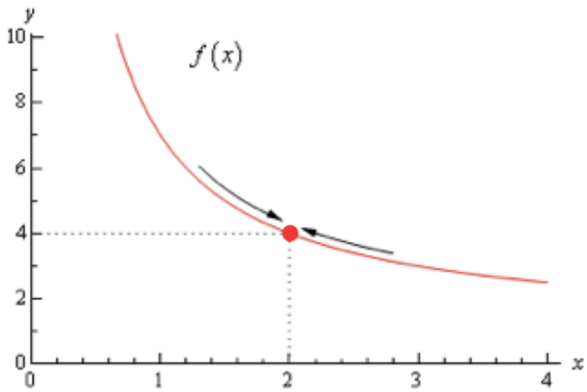
x $(x \to 1^-)$	f(x)
0.8	2.6
0.9	2.8
0.99	2.98
0.999	2.998
1	

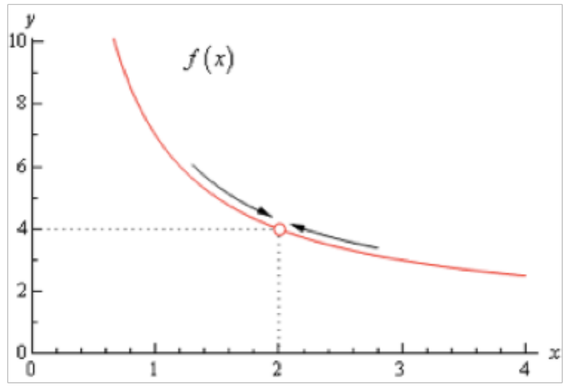


Position (x)	Speed in mph S(x)
3	5
3.5	5.7
3.9	5.89
3.99	5.985
3.999	5.996

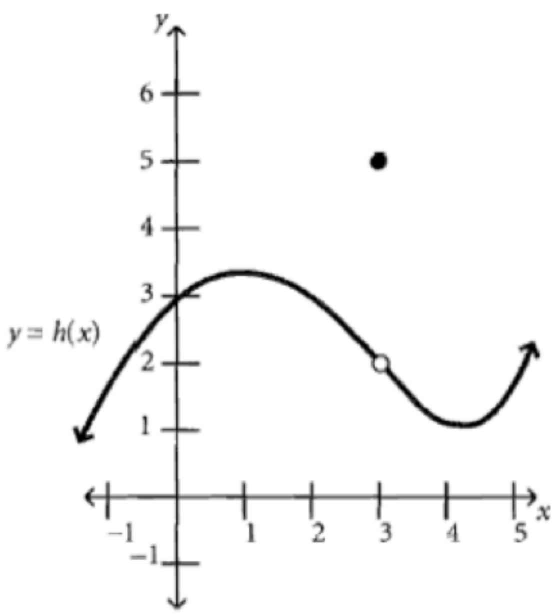
If we want to know the speed of the wind produced by the fan as x gets close to 4, the exact position of the fan. We would measure the wind speed as we get closer and closer to x = 4. The problem is we can not get to the exact position x = 4 or we would risk bodily injury. As a result, we could use the data listed above to conclude that the wind speed of the fan at x = 4 is approaching 6 mph. This is the idea of a limit. Mathematically we would write this as $\lim_{x \to 4} S(x) = 6$

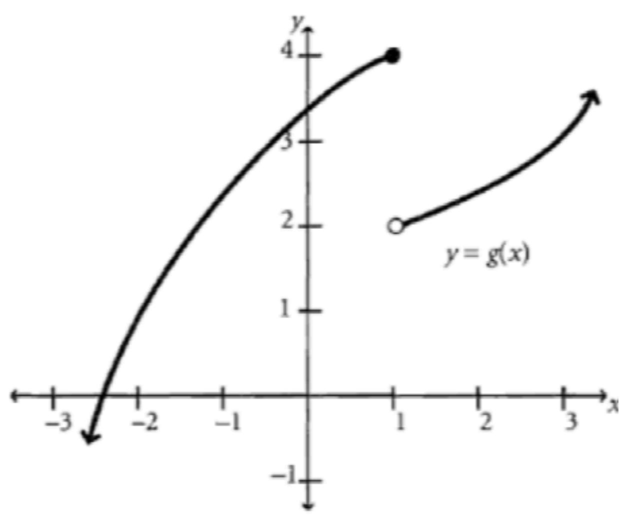
Let's look at more examples:





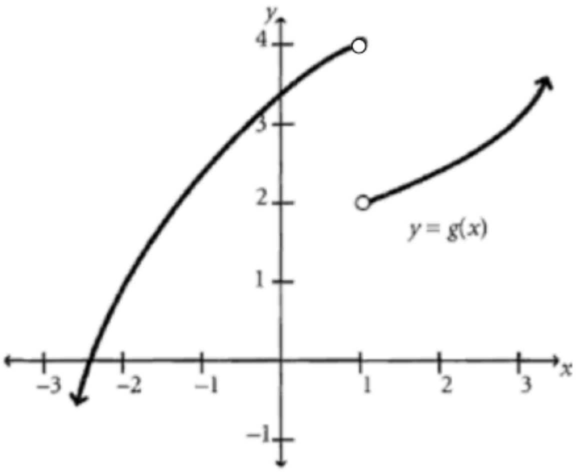
$\lim_{x \to 2} f(x) =$ $\lim_{x \to 2} f(x) =$





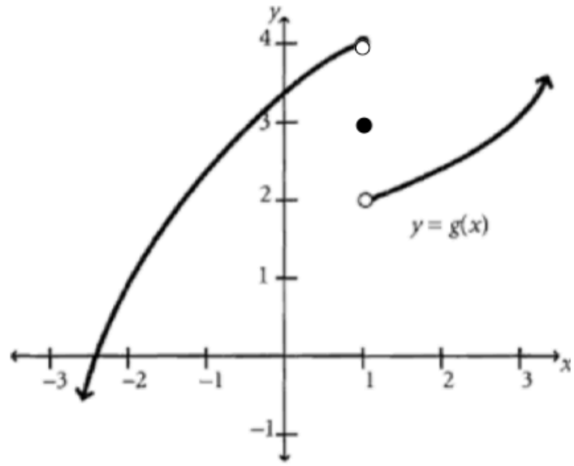
$\lim_{x \to 3} h(x) =$ $\lim_{x \to 1} g(x) =$

$h(3) =$ Why?



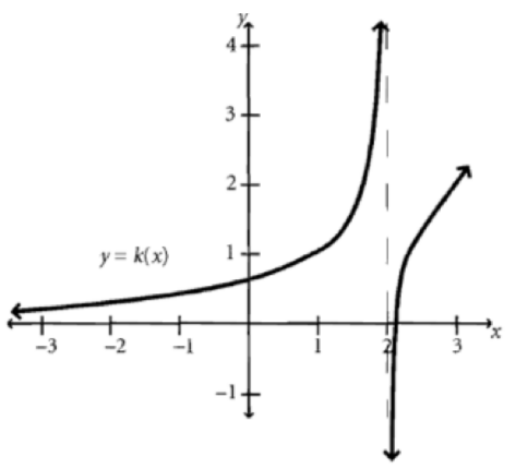
$\lim_{x \to 1} g(x) =$

Why?



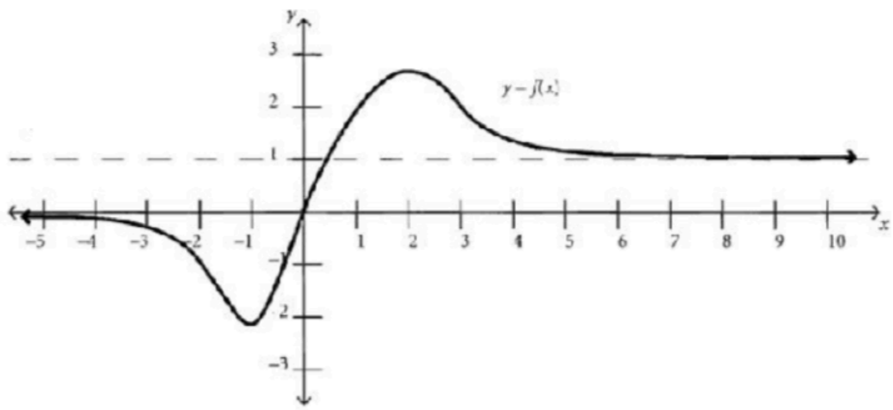
$\lim_{x \to 1} g(x) =$

$g(1) =$



$\lim_{x \to 2} k(x) =$

Why?



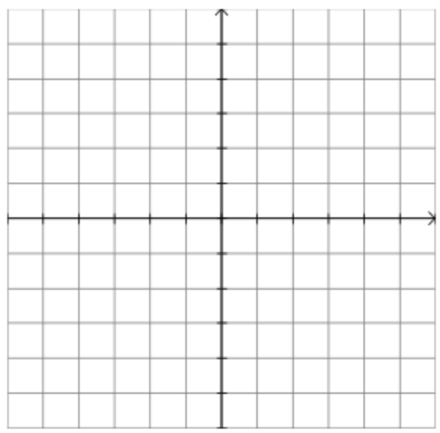
$\lim_{x \to -\infty} f(x) =$

$\lim_{x \to +\infty} f(x) =$

2. One-sided and Two-sided Limits

Example 3: Use a graph and input-output table to find each of the limits for the function $F(x)=\dfrac{1}{x}$, if they exist:

a) $\lim\limits_{x\to 0} F(x)$



b) $\lim\limits_{x\to 0} F(x)$

x	f(x)
0	

In order for a limit to exist, ____

We use the notation:

$\lim\limits_{x\to a^+} f(x)$ to indicate the limit from the ________

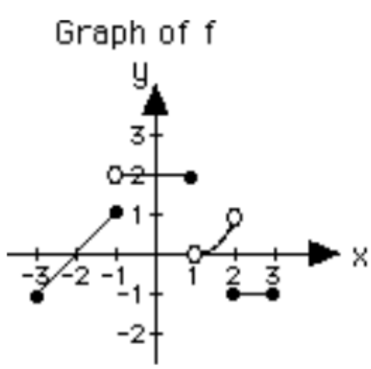
$\lim\limits_{x\to a^-} f(x)$ to indicate the limit from the ________

Theorem – A function $f(x)$ has a limit as x approaches "a" if and only if the right-hand and left-hand limits at "a" exists and are equal.

$$\lim_{x\to a} f(x)=L \text{ iff } \lim_{x\to a^+} f(x)=L=\lim_{x\to a^-} f(x)$$

Example 4: Find the limits.

A.



a) $\lim\limits_{x\to 2+} f(x)=$

b) $\lim\limits_{x\to 2-} f(x)=$

c) $\lim\limits_{x\to 2} f(x)=$

d) $f(2)=$

e) $\lim\limits_{x\to 0} f(x)=$

f) $\lim\limits_{x\to -1+} f(x)=$

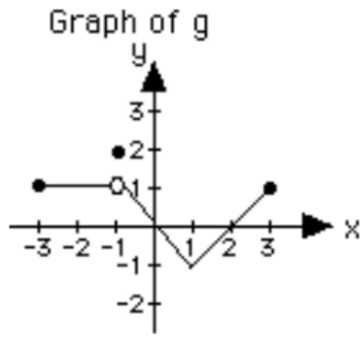
g) $\lim\limits_{x\to -1-} f(x)=$

h) $\lim\limits_{x\to -1} f(x)=$

i) $f(-1)=$

j) $\lim\limits_{x\to 1} f(x)=$

B.



a) $\lim\limits_{x\to 1+} g(x) =$

b) $\lim\limits_{x\to 1-} g(x) =$

c) $\lim\limits_{x\to 1} g(x) =$

d) $\lim\limits_{x\to 0} g(x) =$

e) $\lim\limits_{x\to -1+} g(x) =$

f) $\lim\limits_{x\to -1-} g(x) =$

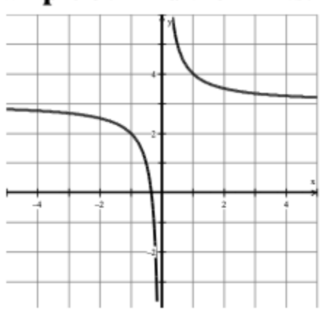
g) $\lim\limits_{x\to -1} g(x) =$

h) $g(-1) =$

3. Infinite Limits and Vertical Asymptotes

If the values of $f(x)$ increase/decrease indefinitely as x approaches "a" from the right or left, the limit does not exist. However, we will write ∞ or $-\infty$.

Example 5: Find the limits.



a) $\lim\limits_{x\to 0+} f(x) =$

b) $\lim\limits_{x\to 0-} f(x) =$

c) $\lim\limits_{x\to 0} f(x) =$

Definition: Vertical Asymptote

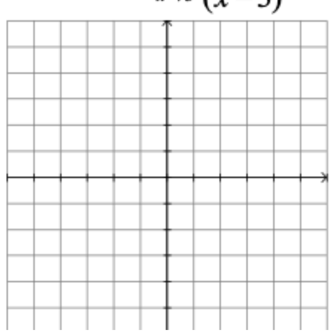
The line $x = a$ is a ***vertical asymptote*** of the graph of a function $y = f(x)$ if either

If both one-sided limits are the same, then we can write:

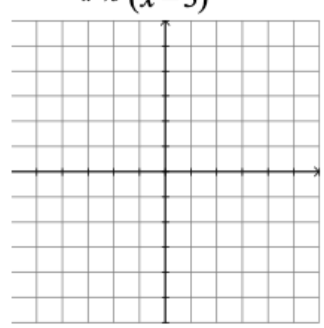
$$\lim_{x\to a} f(x) = +\infty \qquad\qquad \lim_{x\to a} f(x) = -\infty$$

Example 6

a) Graph and find $\lim\limits_{x\to 3} \dfrac{1}{(x-3)^2}$.

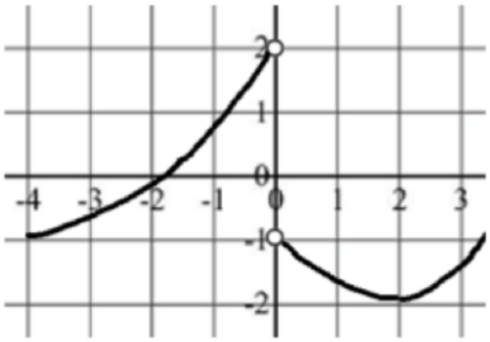


b) Graph and find $\lim\limits_{x\to 3} \dfrac{-1}{(x-3)^2}$.



Example 7

Consider the function g defined by the graph below.



$y = g(x)$

Circle the best choice:

1) Find $\lim_{x \to -1^-} g(x+1)$

(A) -4 (B) -3 (C) -2 (D) -1 (E) 0 (F) 1 (G) 2 (H) 3 (I) 4 (J) DNE

2) Find $\lim_{x \to 0^-} g(x^2)$

(A) -4 (B) -3 (C) -2 (D) -1 (E) 0 (F) 1 (G) 2 (H) 3 (I) 4 (J) DNE

3) Find $\lim_{x \to 1^+} \left(g(x-1)\right)^2$

(A) -4 (B) -3 (C) -2 (D) -1 (E) 0 (F) 1 (G) 2 (H) 3 (I) 4 (J) DNE

4. Horizontal Asymptotes

If the limit as x approaches +∞ or -∞ is a constant, c, then y = c is a horizontal asymptote.
The degree of the numerator must be less than or equal to the degree of the denominator.
You must check for crossing with horizontal asymptotes.

Examples:

1. $f(x) = \frac{x^2-4}{x^3+1}$

$\lim_{x \to \infty} \frac{x^2-4}{x^3+1} =$

Horizontal Asymptote(s):

2. $f(x) = \frac{4x^2+2}{x^2-9}$

$\lim_{x \to \infty} \frac{4x^2+2}{x^2-9} =$

Horizontal Asymptote(s):

*To check for cross point(s), set the function = to the Limit, then solve for x.
The cross point will be (x, L).

Check for crossing: Check for crossing:

You Try:

a) $f(x) = \frac{x}{x^2-9}$

HA:

b) $f(x) = \frac{(x-3)^2}{(x^2-5x)}$

HA:

*To check for cross point(s), set the function = to the Limit, then solve for x.
The cross point will be (x, L).

Check for crossing: Check for crossing:

*Horizontal asymptotes are also known as "end behaviour" since they describe what is happening at both ends of a function.

**A rational function will have a horizontal asymptote when the degree of the numerator is either less than or equal to the degree of the denominator.

5. Oblique or Slant Asymptotes

Oblique Asymptotes

An *oblique asymptote*, or *slant asymptote*, is a linear asymptote that is neither vertical nor horizontal.

For rational functions involving polynomials, oblique asymptotes occur when the numerator has a degree one greater than the denominator.

For example, the rational function $y = \dfrac{x^2+3x}{x-2}$ has an oblique asymptote with equation $y = x+5$, whereas the rational function $y = \dfrac{x}{x^2-4}$ does not have an oblique asymptote.

Rational functions involving polynomials may contain a horizontal asymptote or an oblique asymptote, but not both.

The equation of an oblique asymptote may be found using long or ~~synthetic~~ division.
Long

Oblique Asymptotes

Example

Determine the equation of any oblique asymptotes for $f(x) = \dfrac{x^2+5x-4}{x+3}$.

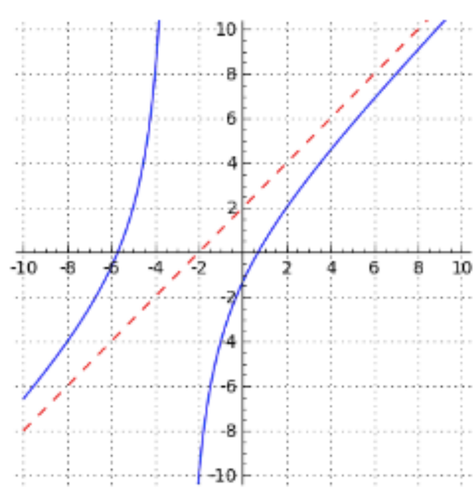
Use $x=-3$ and ~~synthetic~~ Long division to find the quotient, which is the equation of the asymptote.

$$
\begin{array}{r|l}
 & \quad x \;+\; 2 \\
\hline
x+3 & x^2+5x-4 \\
 & -x^2 \cancel{+} 3x \\
\hline
 & \qquad 2x-4 \\
 & \quad -2x \cancel{+} 6 \\
\hline
 & \qquad \mathbf{-10} \quad \textbf{\textit{Remainder}}
\end{array}
$$

Therefore, $f(x)$ has an oblique asymptote with equation $y = x+2$.

Oblique Asymptotes

A graph of $f(x)$ is below.



You Try:

Identify the oblique or slant asymptotes of each.

a) $f(x) = \dfrac{2x^2 + 1}{x}$

SA = ___________________

b) $f(x) = \dfrac{1 - x^2}{x}$

SA = ___________________

c) $f(x) = \dfrac{2x^2 - 5x + 5}{x - 2}$

SA = ___________________

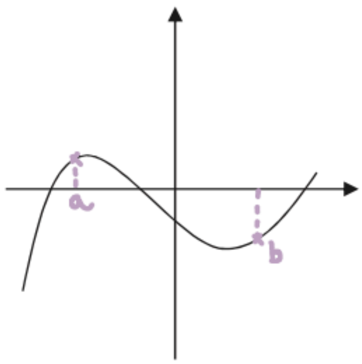
d) $f(x) = \dfrac{2x^3 - x^2 - 2x + 1}{x^2 + 3x + 2}$

SA = ___________________

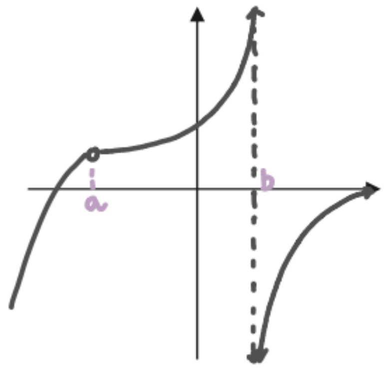
CONTINUITY

A **continuous function**, in everyday language, is a function whose graph has no broken parts. In other words, you can draw its graph without lifting your pencil.

Continuous



Discontinuous



A. Continuity at a Number

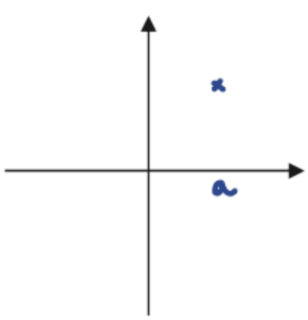
> **1 DEFINITION** A function f is **continuous at a number a** if
>
> $$\lim_{x \to a} f(x) = f(a)$$

Notice that Definition 1 implicitly requires three things if f is continuous at a:

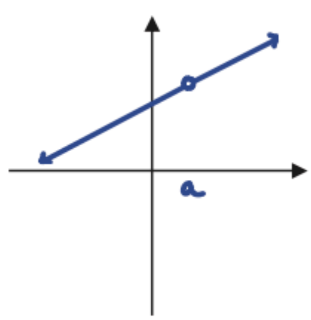
1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

What does this really mean?

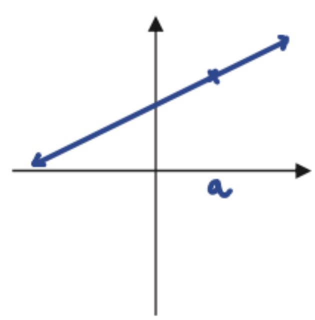
Condition 1:



Condition 2:



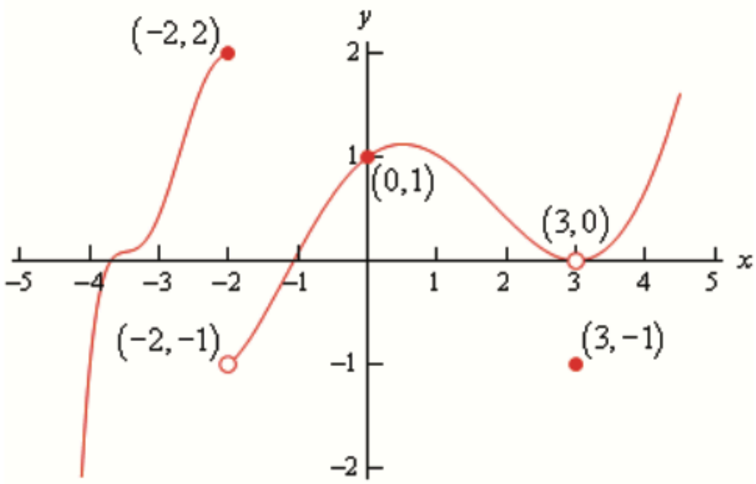
Condition 3:



Therefore, to show that a function is continuous at $x = a$, we must check that all 3 conditions are true. If any one of the conditions is not met, the function is NOT continuous (is discontinuous).

Example 1:

Given the graph of $f(x)$, shown below, determine if $f(x)$ is continuous at $x=-2,\ 0,$ and 3.



Example 2: Show that $f(x)$ is discontinuous at $x=2$, where $f(x)=\dfrac{x^2-x-2}{x-2}$.

> To show that a function is discontinuous, start by looking for possible **points of discontinuity** (are there any values of $f(a)$ that are undefined).

Example 3: Determine where the function is not continuous: $h(t)=\dfrac{4t+10}{t^2-2t-15}$

Example 4: Discuss the continuity of $f(x)$. If discontinuous, identify points of discontinuity.

a. $f(x)=x^2+\sqrt{7-x}$ at $x=4$

b. $f(x)=\begin{cases} x+1 & ,x\le 1 \\ \dfrac{1}{x} & ,1<x<3 \\ \sqrt{x-3} & ,x\ge 3 \end{cases}$ at $x=1,\ 3$

c. $f(x)=\begin{cases} \dfrac{x^2-x-2}{x-2} & ,x\ne 2 \\ & ,x=2 \end{cases}$ at $x=2$

B. Types of Discontinuity

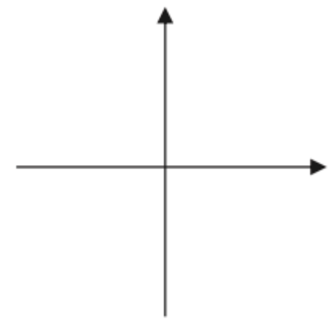
The different types of discontinuity are listed below:

1. **Essential** (or **Infinite**) discontinuity

$\lim_{x \to a^-} f(x) = \pm\infty$

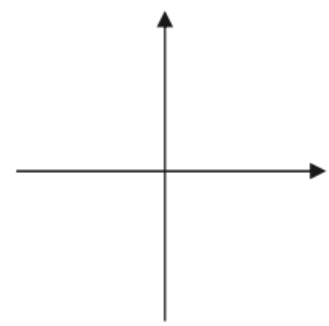
$\lim_{x \to a^+} f(x) = \pm\infty$

$\lim_{x \to a} f(x) = \pm\infty$



2. **Jump** discontinuity

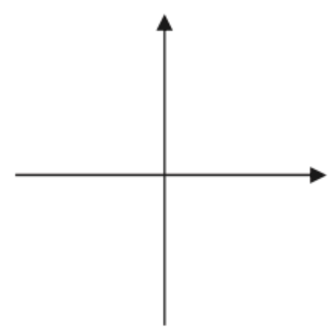
$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$



3. **Point** discontinuity

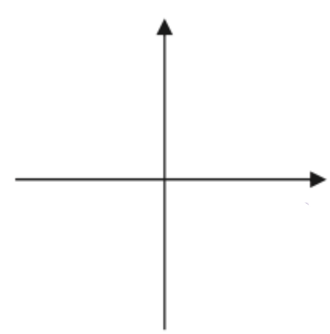
$\lim_{x \to a} f(x) \neq f(a)$

$f(a)$ is undefined.



4. **Removable** discontinuity

$f(a)$ is defined.



C. Continuity on an interval

Definition

A function $f(x)$ is **continuous on an interval** if it is continuous at every number in the interval.

Theorem

The following types of functions are continuous at every number in their domains:

polynomials, rational functions, root functions,
trigonometric functions, inverse trigonometric functions,
exponential functions, logarithmic functions

Example 5: Show that the function is continuous on the given interval.

$f(x) = 2\sqrt{3-x}, \ (-\infty, 3]$

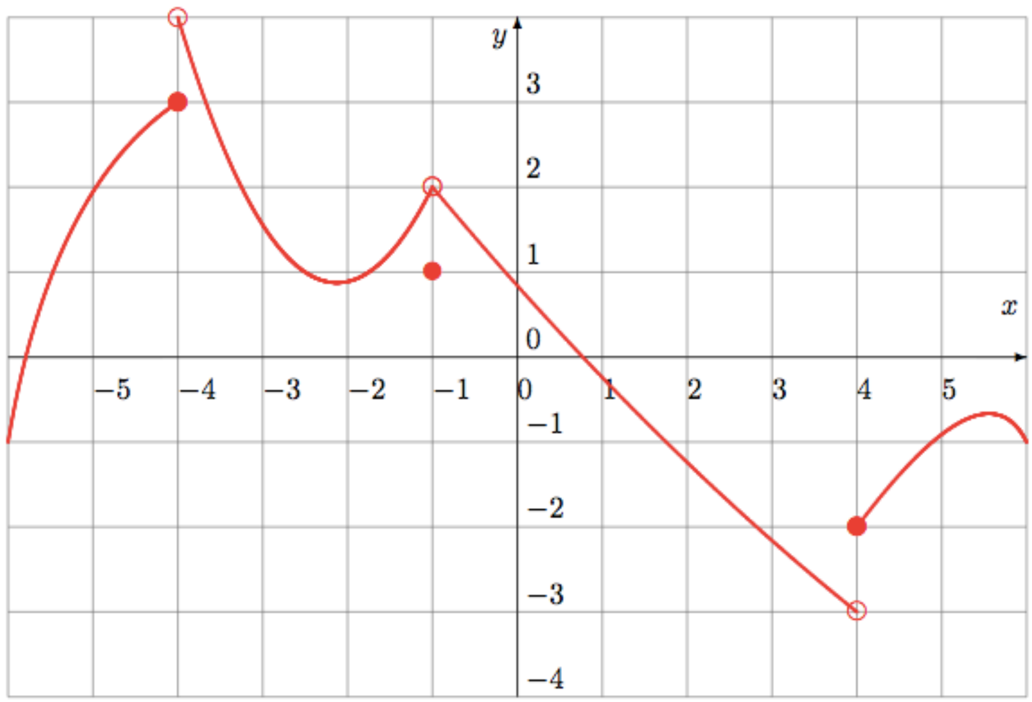
Example 6: Confirm the following functions are continuous over an interval.

$f(x) = \frac{x-5}{x+1}$

$f(x) = ln(2x-1)$

3.1 - PRACTICE QUESTIONS

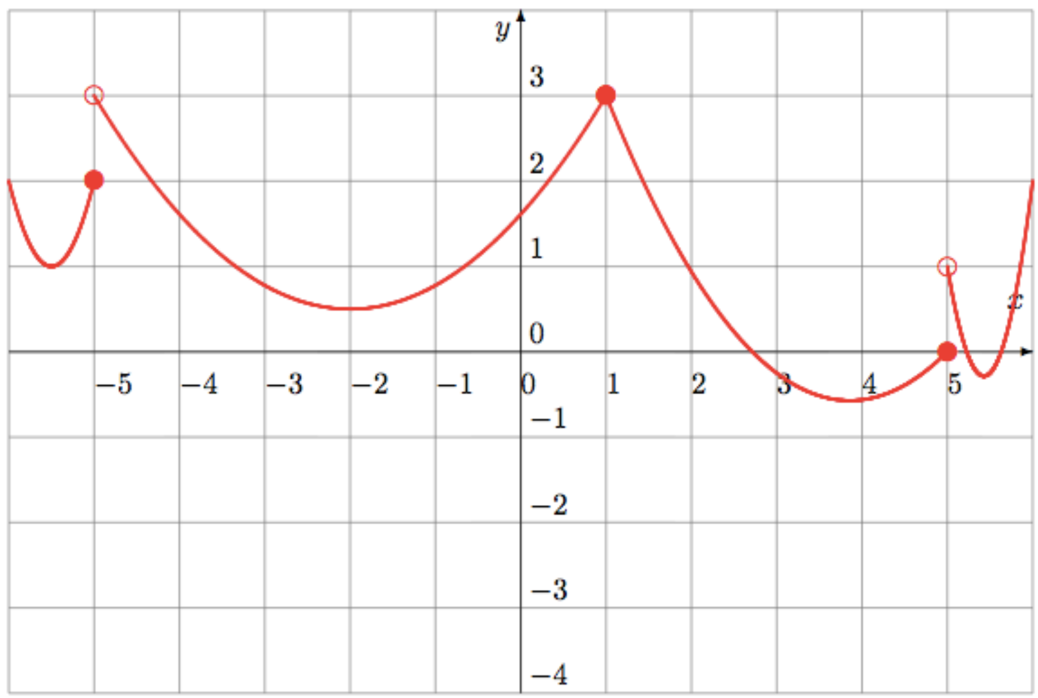
1. Consider the following function defined by its graph:



Find the following limits:

$a)\ \lim_{x \to -1^-} f(x)$ $\quad b)\ \lim_{x \to -1^+} f(x)$ $\quad c)\ \lim_{x \to -1} f(x)$ $\quad d)\ \lim_{x \to -4} f(x)$ $\quad e)\ \lim_{x \to 4} f(x)$

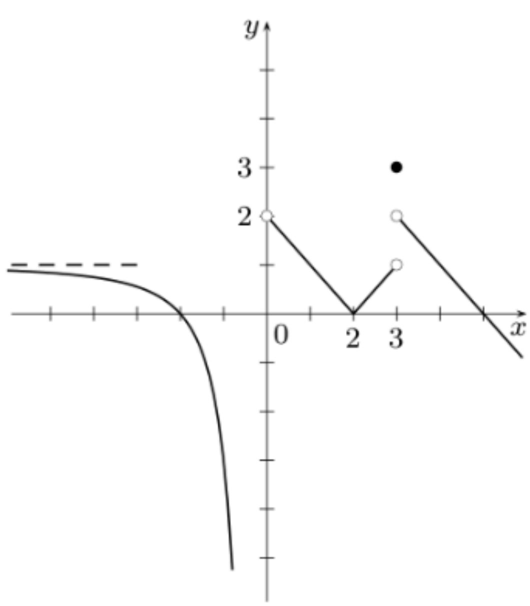
2. Consider the following function defined by its graph:



Find the following limits:

$a)\ \lim_{x\to 1^-} f(x)$ $\quad b)\ \lim_{x\to 1^+} f(x)$ $\quad c)\ \lim_{x\to 1} f(x)$ $\quad d)\ \lim_{x\to -5} f(x)$ $\quad e)\ \lim_{x\to 5} f(x)$

3. *Use the graph of the function f(x) to answer each question.*
Use $\infty, -\infty$ *or* DNE *where appropriate.*



(a) $f(0) =$

(b) $f(2) =$

(c) $f(3) =$

(d) $\lim_{x\to 0^-} f(x) =$

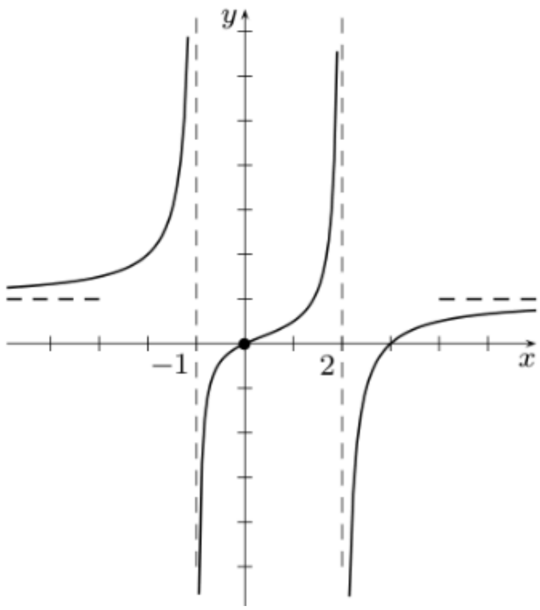
(e) $\lim_{x\to 0} f(x) =$

(f) $\lim_{x\to 3^+} f(x) =$

(g) $\lim_{x\to 3} f(x) =$

(h) $\lim_{x\to -\infty} f(x) =$

4. *Use the graph of the function $f(x)$ to answer each question.*
Use $\infty, -\infty$ or DNE where appropriate.



(a) $f(0) =$

(b) $f(2) =$

(c) $f(3) =$

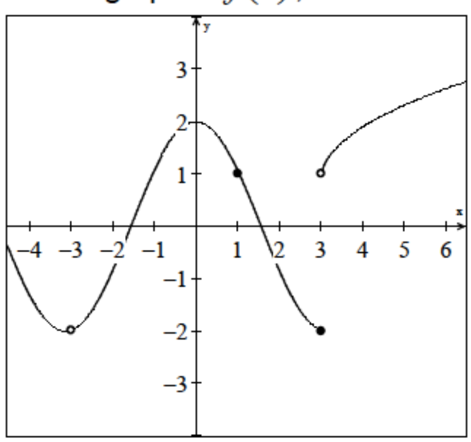
(d) $\lim_{x \to -1} f(x) =$

(e) $\lim_{x \to 0} f(x) =$

(f) $\lim_{x \to 2^+} f(x) =$

(g) $\lim_{x \to \infty} f(x) =$

5. Given the graph of $f(x)$, determine the following.



a) $\lim_{x \to -3^-} f(x)$ b) $\lim_{x \to -3^+} f(x)$ c) $\lim_{x \to -3} f(x)$

d) $\lim_{x \to 1^+} f(x)$ e) $\lim_{x \to 1^-} f(x)$ f) $\lim_{x \to 1} f(x)$

g) $\lim_{x \to 3^-} f(x)$ h) $\lim_{x \to 3^+} f(x)$ i) $\lim_{x \to 3} f(x)$

j) $f(-3)$ k) $f(1)$ l) $f(3)$

6. Identify the points of discontinuity, holes, vertical asymptotes, and horizontal asymptote of each.

a) $f(x) = -\dfrac{4}{x^2 - 3x}$

Disc =	VA =
Holes =	HA =

b) $f(x) = \dfrac{x-4}{-4x-16}$

Disc =	VA =
Holes =	HA =

c) $f(x) = \dfrac{3x^2 - 12x}{x^2 - 2x - 3}$

Disc =	VA =
Holes =	HA =

d) $f(x) = \dfrac{x^2 + x - 6}{-4x^2 - 16x - 12}$

Disc =	VA =
Holes =	HA =

7. MULTIPLE CHOICE. Find the slant asymptote, if any, of the graph of the rational function.

1) $f(x) = \frac{x^2 + 3x - 6}{x - 3}$

A) $y = x + 6$
B) $y = x$
C) $y = x + 3$
D) no slant asymptote

2) $f(x) = \frac{x^2 - 4x + 9}{x + 5}$

A) $y = x - 9$
B) $x = y + 4$
C) $y = x + 13$
D) no slant asymptote

3) $f(x) = \frac{x^2 - 8x + 9}{x + 4}$

A) $y = x - 12$
B) $x = y + 8$
C) $y = x + 17$
D) no slant asymptote

4) $f(x) = \frac{x^2 - 2x + 2}{x + 2}$

A) $x = y + 2$
B) $y = x - 4$
C) $y = x + 4$
D) no slant asymptote

8. Identify the horizontal asymptotes of each using limits to ∞.

$a)\ \lim_{x\to\infty} \frac{3x + 5}{x - 4}$

$b)\ \lim_{t\to\infty} \frac{t^2 + 2}{t^3 + t^2 - 1}$

$c)\ \lim_{t\to\infty} \frac{t^3 + t^2 - 1}{t^2 + 2}$

$d)\ \lim_{x\to\infty} \frac{x + 2}{\sqrt{9x^2 + 1}}$

9. Calculate the following limits.

$\lim_{x \to \infty} f(x) =$ ________

$\lim_{x \to -\infty} f(x) =$ ________

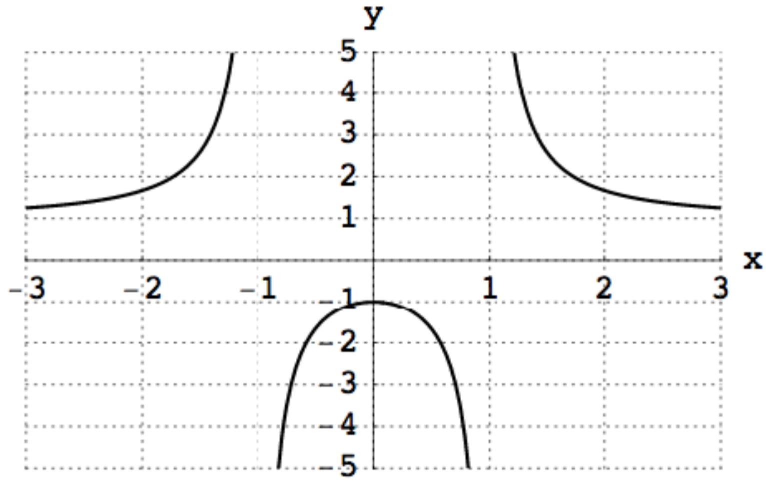
$\lim_{x \to -1^+} f(x) =$ ________

$\lim_{x \to -1^-} f(x) =$ ________

$\lim_{x \to 1^+} f(x) =$ ________

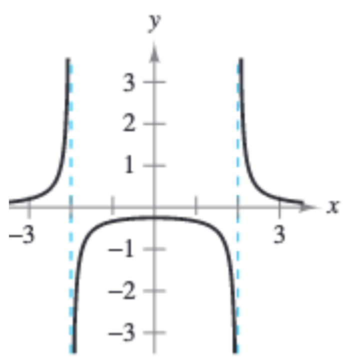
$\lim_{x \to 1^-} f(x) =$ ________

$$f(x) = \frac{x^2+1}{x^2-1}$$

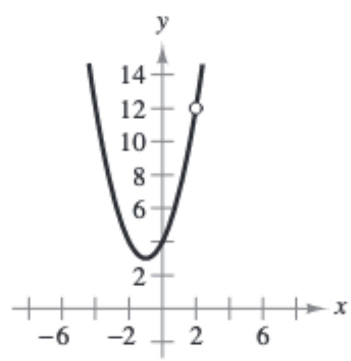


10. Describe the interval(s) on which the function is continuous on the entire real line.

a) $f(x) = \dfrac{1}{x^2 - 4}$



b) $f(x) = \dfrac{x^3 - 8}{x - 2}$

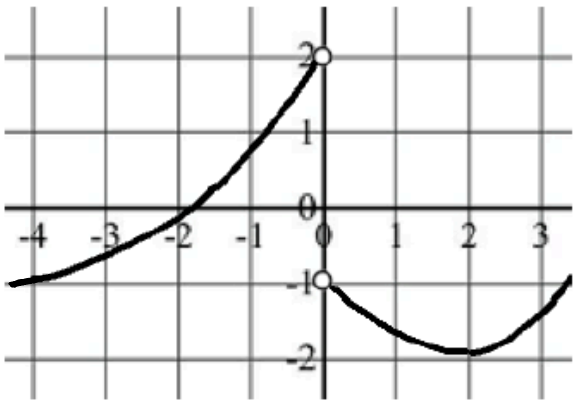


11. Describe the continuity of the function on the closed interval. If there are any discontinuities, determine whether they are removable.

Function	*Interval*
$a)\ \ f(x) = \dfrac{1}{x-2}$	$[1, 4]$
$b)\ \ f(x) = \dfrac{x}{x^2 - 4x + 3}$	$[0, 4]$
$c)\ \ f(x) = \dfrac{x^2 - 16}{x - 4}$	$[1, 5]$

EXTENDED QUESTIONS

Consider the function g defined by the graph below.



$y = g(x)$

Circle the best choice:

1) Find $\lim_{x \to -1^+} g(x+1)$

(A) -4 (B) -3 (C) -2 (D) -1 (E) 0 (F) 1 (G) 2 (H) 3 (I) 4 (j) DNE

2) Find $\lim_{x \to 2^-} g(-x^2)$

(A) -4 (B) -3 (C) -2 (D) -1 (E) 0 (F) 1 (G) 2 (H) 3 (I) 4 (j) DNE

3) Find $\lim_{x \to 1^-} \left(g(x-1)\right)^2$

(A) -4 (B) -3 (C) -2 (D) -1 (E) 0 (F) 1 (G) 2 (H) 3 (I) 4 (j) DNE

4.

Sketch the graph of a function that satisfies all of the following properties at once.

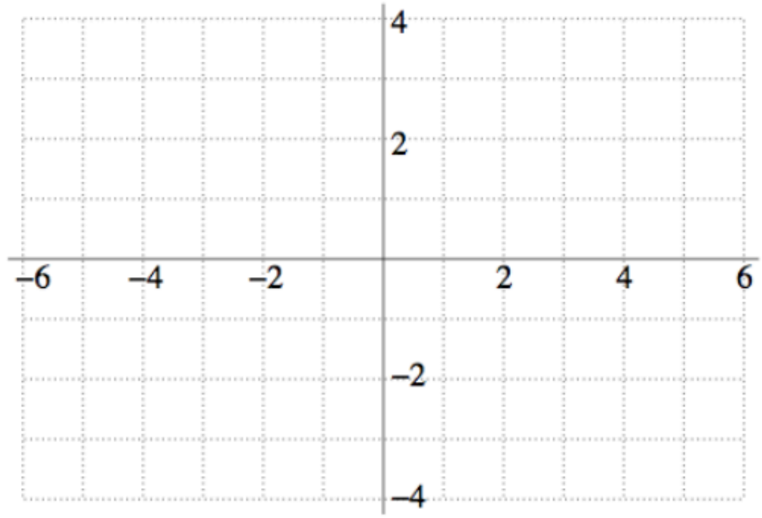
(a) $\lim_{x \to -2^+} f(x) = \infty$

(b) $\lim_{x \to -2^-} f(x) = -\infty$

(c) $\lim_{x \to \infty} f(x) = \infty$

(d) $\lim_{x \to -\infty} f(x) = 0$

(e) $\lim_{x \to 3} f(x) = -\infty$

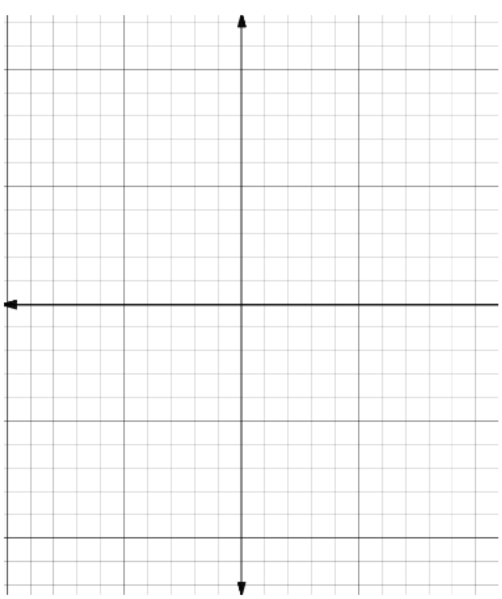


5. Calculate the following limits. **Cannot use L'H

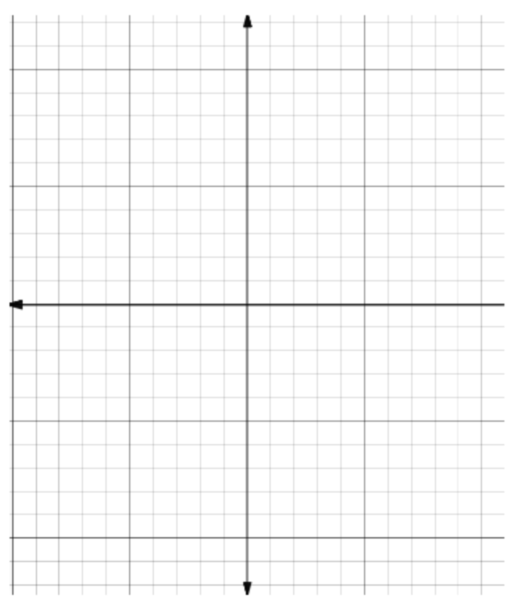
(a) $\lim_{x \to \infty} \dfrac{7x^3 + 4x}{2x^3 - x^2 + 3}$

(d) $\lim_{t \to \pi^+} \csc t$

Sketch the above function.



Sketch the above function.



6.

$$\lim_{x\to\infty}\frac{(2x-1)(3-x)}{(x-1)(x+3)} \text{ is}$$

(A) -3 (B) -2 (C) 2 (D) 3 (E) nonexistent

7.

$$\lim_{x\to 0}\frac{5x^4+8x^2}{3x^4-16x^2} \text{ is}$$

(A) $-\frac{1}{2}$ (B) 0 (C) 1 (D) $\frac{5}{3}+1$ (E) nonexistent

8.
$$f(x)=\begin{cases}\dfrac{x^2-4}{x-2} & \text{if } x\neq 2\\ 1 & \text{if } x=2\end{cases}$$

Let f be the function defined above. Which of the following statements about f are true?

I. f has a limit at $x=2$.

II. f is continuous at $x=2$.

III. f is differentiable at $x=2$.

(A) I only

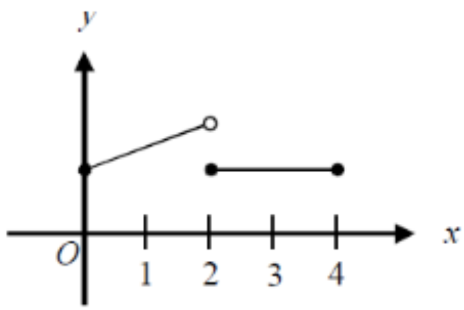
(B) II only

(C) III only

(D) I and II only

(E) I, II, and III

9.



Graph of f

The figure above shows the graph of a function f with domain $0 \le x \le 4$. Which of the following statements are true?

I. $\lim\limits_{x\to 2^-} f(x)$ exists.

II. $\lim\limits_{x\to 2^+} f(x)$ exists.

III. $\lim\limits_{x\to 2} f(x)$ exists.

(A) I only (B) II only (C) I and II only (D) I and III only (E) I, II, and III

10.

For $x \ge 0$, the horizontal line $y = 2$ is an asymptote for the graph of the function f. Which of the following statements must be true?

(A) $f(0) = 2$

(B) $f(x) \ne 2$ for all $x \ge 0$

(C) $f(2)$ is undefined.

(D) $\lim\limits_{x\to 2} f(x) = \infty$

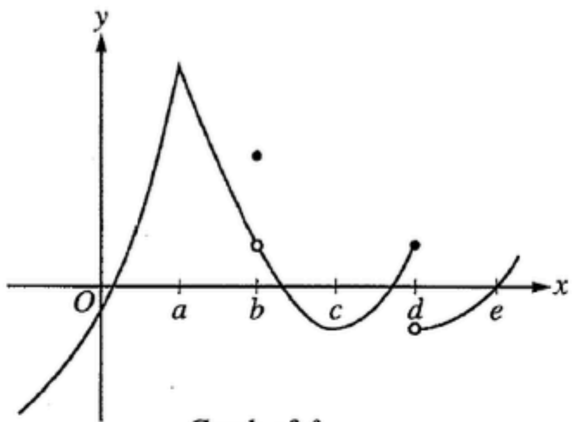
(E) $\lim\limits_{x\to\infty} f(x) = 2$

11.

$$\lim_{x\to\infty} \frac{x^3 - 2x^2 + 3x - 4}{4x^3 - 3x^2 + 2x - 1} =$$

(A) 4 (B) 1 (C) $\frac{1}{4}$ (D) 0 (E) -1

12.



Graph of f

. The graph of a function f is shown above. At which value of x is f continuous, but not differentiable?

(A) a (B) b (C) c (D) d (E) e

13.

$$f(x) = \begin{cases} x + 2 & \text{if } x \le 3 \\ 4x - 7 & \text{if } x > 3 \end{cases}$$

Let f be the function given above. Which of the following statements are true about f ?

I. $\lim_{x \to 3} f(x)$ exists.

II. f is continuous at $x = 3$.

III. f is differentiable at $x = 3$.

(A) None
(B) I only
(C) II only
(D) I and II only
(E) I, II, and III

14.

Let f be a function defined by $f(x) = \begin{cases} 1 - 2\sin x & \text{for } x \le 0 \\ e^{-4x} & \text{for } x > 0. \end{cases}$

(a) Show that f is continuous at $x = 0$.

15.

A 12,000-liter tank of water is filled to capacity. At time $t = 0$, water begins to drain out of the tank at a rate modeled by $r(t)$, measured in liters per hour, where r is given by the piecewise-defined function

$$r(t) = \begin{cases} \dfrac{600t}{t+3} & \text{for } 0 \le t \le 5 \\ 1000e^{-0.2t} & \text{for } t > 5 \end{cases}$$

(a) Is r continuous at $t = 5$? Show the work that leads to your answer.

16.

Let f be the function given by $f(x) = \dfrac{\ln x}{x}$ for all $x > 0$. The derivative of f is given by $f'(x) = \dfrac{1 - \ln x}{x^2}$.

Find $\lim\limits_{x \to 0^+} f(x)$.

17. Let f be the function defined by

$$f(x) = \begin{cases} \sqrt{x+1} & \text{for } 0 \le x \le 3 \\ 5 - x & \text{for } 3 < x \le 5. \end{cases}$$

(a) Is f continuous at $x = 3$? Explain why or why not.

18.

Given the function $f(x)=\dfrac{x^3+2x^2-3x}{3x^2+3x-6}$.

(a) What are the zeros of $f(x)$?

(b) What are the vertical asymptotes of $f(x)$?

(c) The end behavior model of $f(x)$ is the function $g(x)$. What is $g(x)$?

(d) What is $\lim_{x\to\infty} f(x)$? What is $\lim_{x\to\infty}\dfrac{f(x)}{g(x)}$?

3.2 - Evaluating Limits

Types of Limits we'll evaluate:

$\lim_{x \to a} f(x)$ $\lim_{x \to \pm\infty} f(x)$ $\lim_{x \to 1^+} f(x)$ *(one-sided limits)*

General Rule with the above problems:

1. ***Plug and Chug*** **[Substitution]**

2. ***However, you may have to use one of the following Tricks:***
   - ***a) L'Hopital Rule***
   - ***b) Factor***
   - ***c) Conjugate***
   - ***d) Complex Fractions***
   - ***e) U-Substitution***

Example:

a) $\lim_{x \to 3} 3x - 1 =$ b) $\lim_{x \to \sqrt{4}} (x^2 - 1)^2 =$

You Try:

a) $\lim_{x \to -1} 2x^2 + 1 =$ b) $\lim_{x \to 3} \frac{-x}{x^2 - 6} =$ c) $\lim_{x \to 9} \left(\sqrt{x} + \frac{3}{\sqrt{x}}\right)^2 =$

Evaluating Limits with Properties

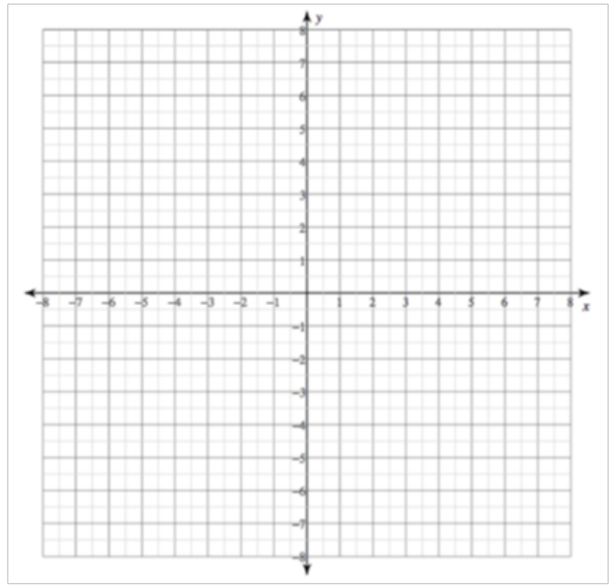
If $\lim_{x \to 2} f(x) = 5$, use the properties of limits to evaluate the limit: $\lim_{x \to 2} \sqrt{2f(x) - x^2} =$

Evaluating One-Sided Limits

Example:

$$f(x)=\begin{cases} x, & \text{if } x \le 1 \\ -x+3, & \text{if } x>1 \end{cases} \quad \lim_{x\to 1} f(x)$$

Algebraically *Visually*



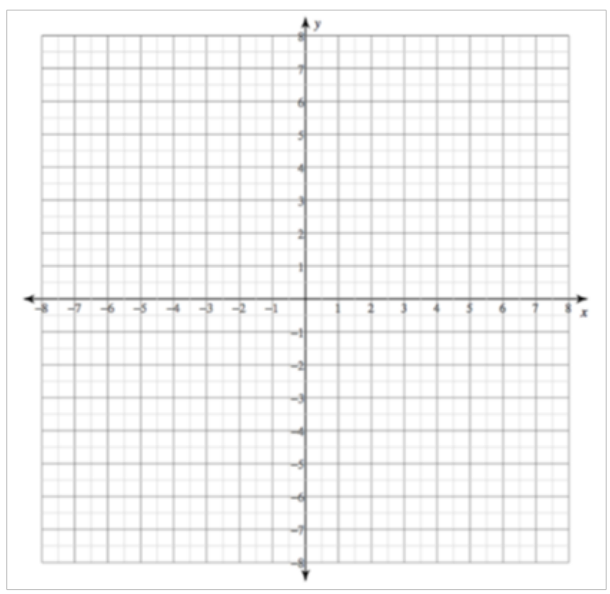
You Try:

$$f(x)=\begin{cases} x+2\ ; & x>1 \\ x^2-1\ ; & x\le 1 \end{cases}$$

Evaluate the following limits, if they exist.

$\lim_{x\to 1^+} f(x) =$ $\lim_{x\to 1^-} f(x) =$ $\lim_{x\to 1} f(x) =$

Algebraically *Visually*



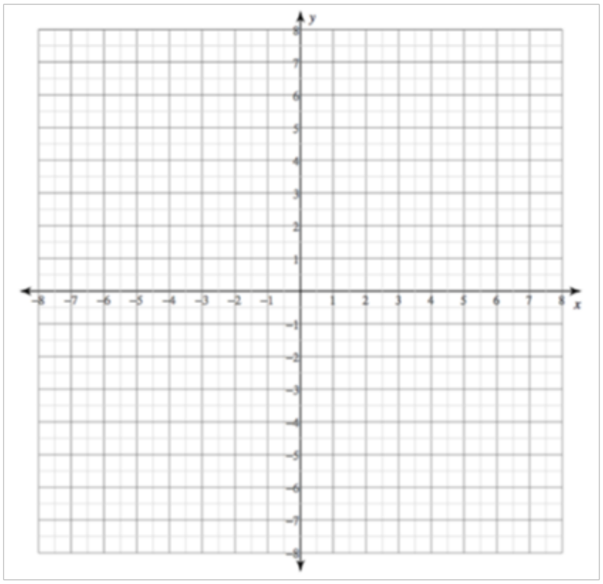
Evaluating Radical Functions **Watch out for One-Sided Limits

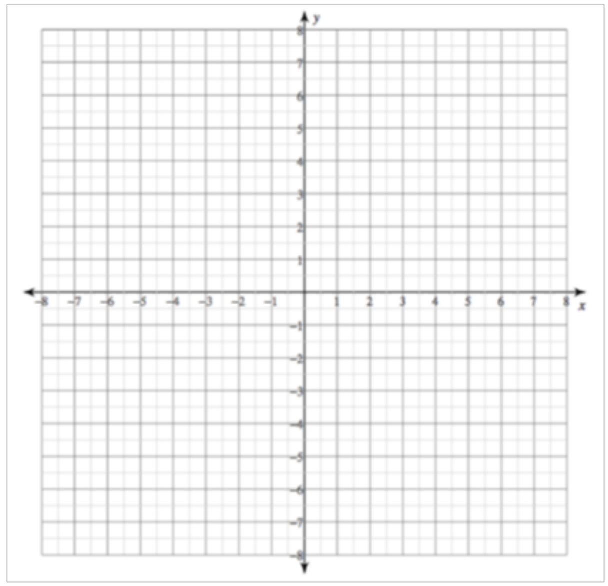
Examples:

a) $\lim_{x \to 0} \sqrt{x} =$

b) $\lim_{x \to 2} \sqrt{2 - x} =$

Note: To fully grasp what the limit is, sketch a graph

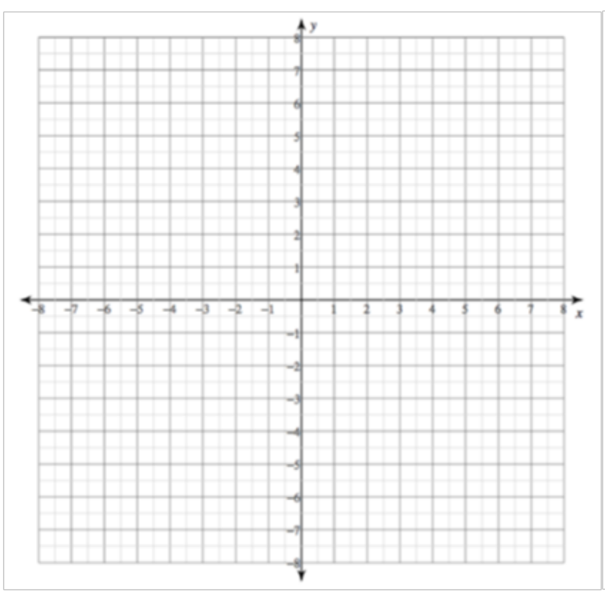


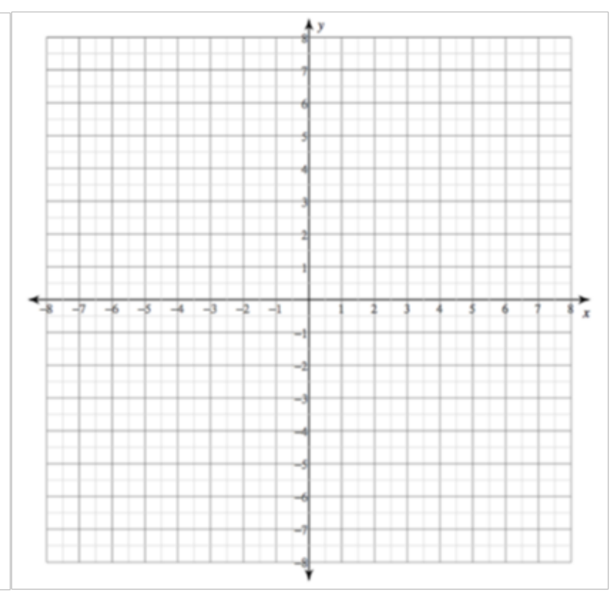


You Try:

$\lim_{x \to -3} \sqrt{x + 3} =$

$\lim_{x \to 1} \sqrt{x + 3}$





Evaluating Limits using "L'Hopital's Rule"

Examples:

$$\lim_{x\to 4}\frac{x^2-16}{x-4} \qquad \lim_{x\to\infty}\frac{4x^2-5x}{1-3x^2}$$

In the first limit if we plugged in $x = 4$ we would get 0/0 and in the second limit if we "plugged" in infinity we would get $\infty/-\infty$. Both of these are called **indeterminate forms**.

L'Hospital's Rule

Suppose that we have one of the following cases:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0} \quad \text{OR} \quad \lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

In these cases we have to take the derivative of the top and bottom separately.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

***Sometimes we will need to apply L'Hospital's Rule more than once.*

Examples:

$$\lim_{x\to 4}\frac{x^2-16}{x-4} \qquad \lim_{x\to\infty}\frac{4x^2-5x}{1-3x^2}$$

You Try:

$$\lim_{x\to 3}\frac{2x-6}{x^2-9} = \qquad\qquad \lim_{x\to 0}\frac{3\sin(2x)}{2x} =$$

Evaluating Limits by "Factoring"

Example:

$$\lim_{x \to 3} \frac{x^2-9}{x-3} =$$

You Try:

$$\lim_{x \to -4} \frac{x^2+4x}{x^2+x-12} =$$

Evaluating Limits by "Rationalizing" or "Conjugating"

Example:

$$\lim_{x \to 3} \frac{\sqrt{3}-\sqrt{x}}{3-x} =$$

You Try:

$$\lim_{x \to 0} \frac{x}{\sqrt{x+9}-3} =$$

3.2 - PRACTICE QUESTIONS

1. *Evaluate each limit:*

$a)\ \lim_{x\to 3} 2x-8$ $\quad b)\ \lim_{x\to -2} x^2-1$ $\quad c)\ \lim_{x\to -2} \left(x^2-2x-3\right)$

$d)\ \lim_{x\to \pi} 2x^2+2x-1$ $\quad e)\ \lim_{x\to \sqrt{2}} \left(x^2-1\right)$ $\quad f)\ \lim_{x\to -\sqrt{2}} x^2-2x-1$

2. *By evaluating one-sided limits, find the indicated limit if it exists:*

$a)\ f(x)=\begin{cases} x+2 & \text{where } x<-1 \\ -x+2 & \text{where } x\ge -1 \end{cases} \quad \lim_{x\to -1} f(x)$ $\qquad b)\ f(x)=\begin{cases} x & \text{where } x\le 1 \\ -x+3 & \text{where } x>1 \end{cases} \quad \lim_{x\to 1} f(x)$

$c)\ g(x)=\begin{cases} -x+4 & \text{where } x\le 2 \\ -2x+6 & \text{where } x>2 \end{cases} \quad \lim_{x\to 2} g(x)$ $\qquad d)\ k(x)=\begin{cases} 4-x^2 & \text{where } x<1 \\ x & \text{where } x\ge 1 \end{cases} \quad \lim_{x\to 1} k(x)$

$e)\ h(x)=\begin{cases} 4x & \text{where } x\ge \frac{1}{2} \\ \frac{1}{x} & \text{where } x<\frac{1}{2} \end{cases} \quad \lim_{x\to \frac{1}{2}} h(x)$ $\qquad f)\ m(x)=\begin{cases} x+3 & \text{where } x<2 \\ \frac{4}{x} & \text{where } x\ge 2 \end{cases} \quad \lim_{x\to 2} m(x)$

3. *Evaluate each of the following limits:*

$a)\ \lim_{x\to 2} \dfrac{3x}{x^2+2}$ $b)\ \lim_{x\to -1} x^4+x^3+x^2$ $c)\ \lim_{x\to 3} \sqrt{x^3+\dfrac{27}{x-1}}$

$d)\ \lim_{x\to 4} \left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)^2$ $e)\ \lim_{x\to 2\pi} \left(x^3+\pi^2 x-5\pi^3\right)$ $f)\ \lim_{x\to a} \dfrac{(x+a)^2}{x^2+a^2}$

$g)\ \lim_{x\to 0} \sqrt{1+\sqrt{1+x}}$ $h)\ \lim_{h\to 0} \dfrac{1}{\sqrt{x}+\sqrt{x+h}}$ $i)\ \lim_{x\to 4} \left(\sqrt{x}+2\right)^3$

4. *If* $\lim_{x\to 2} f(x)=3$*, use the properties of limits to evaluate each of the limits below:*

$a)\ \lim_{x\to 2} \dfrac{x^2+5}{f(x)}$ $b)\ \lim_{x\to 2} \sqrt{\left[f(x)\right]^2+x^4}$ $c)\ \lim_{x\to 2} \sqrt{3f(x)-2x}$

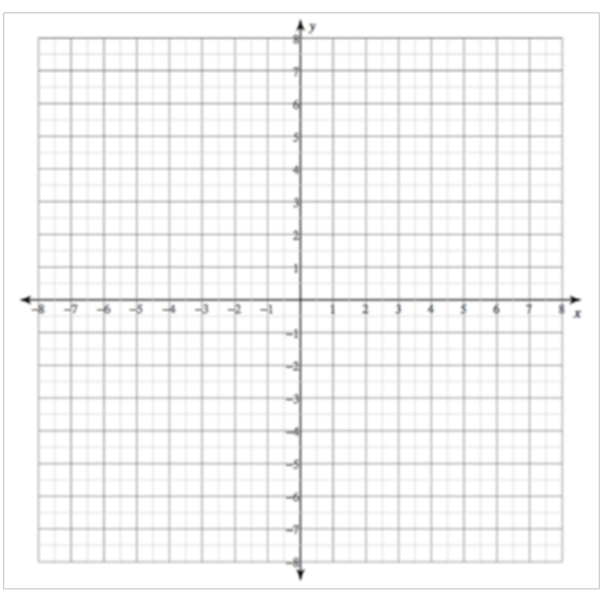
5. *Evaluate each limit.*

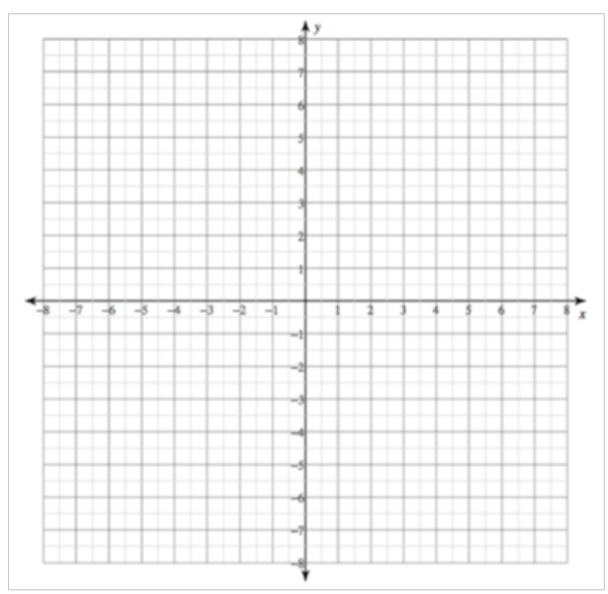
$a) \quad \lim_{x \to 1^+} \frac{x^2 + x + 2}{x - 1}$

$b) \quad \lim_{x \to -3^-} \frac{x + 7}{x + 3}$

$c) \quad \lim_{x \to 2^+} \sqrt{x - 2}$

$d) \quad \lim_{x \to -5^-} \sqrt{x + 5}$





6. *Evaluate each of the following limits using* ***L'HOPITAL'S RULE:***

$a)\ \lim_{x\to 2} \frac{x^2-4}{x-2}$ $b)\ \lim_{x\to -2} \frac{4-x^2}{2+x}$ $c)\ \lim_{x\to 1} \frac{x^3-1}{x-1}$

$d)\ \lim_{x\to -1} \frac{2x^2+5x+3}{x+1}$ $e)\ \lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$ $f)\ \lim_{x\to 0} \frac{\sqrt{4+x}\ -\ 2}{x}$

$g)\ \lim_{x\to 0} \frac{\cos x-1}{x}$ $h)\ \lim_{x\to 0} \frac{3\sin 2x}{2x}$ $i)\ \lim_{x\to 0} \frac{e^x-1}{2e^x-2}$

$j)\ \lim_{x\to 1} \frac{x-1}{\ln x}$ $k)\ \lim_{x\to 0} \frac{\cos x-1}{e^x-1}$ $l)\ \lim_{x\to 1} \frac{2\ln x}{4x^2-4}$

*Evaluate each limit using **L'HOPITAL'S RULE:**:*

7) $\lim_{x \to 1} - \frac{x^2 - 1}{x - 1}$

8) $\lim_{x \to 5} - \frac{x^2 - 5x}{x - 5}$

9) $\lim_{x \to 2} - \frac{x^2 - x - 2}{x - 2}$

10) $\lim_{x \to -5} \frac{x^2 + 3x - 10}{x + 5}$

11) $\lim_{x \to 0} \frac{\frac{1}{-4 + x} + \frac{1}{4}}{x}$

12) $\lim_{x \to -3} \frac{x}{\frac{1}{3 + x} - \frac{1}{3}}$

13) $\lim_{x \to 5} \frac{x - 5}{\sqrt{x + 4} - 3}$

14) $\lim_{x \to 3} \frac{\sqrt{x + 6} - 3}{x - 3}$

15. *Evaluate each limit using factoring and simplifying. Check your answers using L'Hospital's Rule:*

$a)\ \lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1}$ $b)\ \lim\limits_{x \to 2} \dfrac{4 - x^2}{x - 2}$ $c)\ \lim\limits_{x \to -1} \dfrac{x^2 - x - 2}{x^2 + 4x + 3}$

16. *Evaluate each limit by rationalizing the numerator. Check your answers using L'Hospital's Rule:*

$a)\ \lim\limits_{x \to 1} \dfrac{\sqrt{x} - 1}{x - 1}$ $b)\ \lim\limits_{x \to 2} \dfrac{\sqrt{2} - \sqrt{x}}{2 - x}$ $c)\ \lim\limits_{x \to 0} \dfrac{\sqrt{x + 4} - 2}{x}$

***BE SURE TO DO MULTIPLE CHOICE QUESTIONS ONLINE **VERSION 13**
#39 – 49, #184 – 188
https://moodle.sd79.bc.ca/pluginfile.php/64324/mod_resource/content/4/AB%20Calculus%20Version%2013.pdf

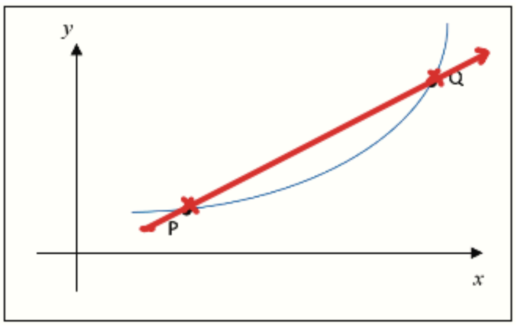
3.3 - DEFINITION OF DERIVATIVE

In the first section of this chapter, we saw that the computation of the slope of a tangent line and the instantaneous rate of change of a function required us to make the secant line very close to the tangent line.

Revisiting this idea, we now define the slope of a tangent line using limit notation.

***A. The definition of derivative Definition** 1.*

Slope of PQ =



Slope of tangent at P =

$f'(x) =$

Example 1 : Use the "Definition of the Derivative" to find $\frac{dy}{dx}$ of $y = 3x + 1$.

Example 2 : Use the "Definition of the Derivative" to find $\frac{dy}{dx}$ of $y = \frac{3}{x}$.

Example 3 : Find $R'(z)$, given $R(z)=\sqrt{5z-8}$ using the definition of derivative.

Example 4 : Use the "Definition of the Derivative", find $g'(t)$ of $g(t)=\dfrac{t}{t+1}$

3.3 - PRACTICE QUESTIONS

1. *Use the **DEFINITION OF THE DERIVATIVE** to find* $\frac{dy}{dx}$*. Check your answers by differentiation.*

$a)\ y = 2x + 5$ $\quad$ $b)\ y = 3 - 2x$ $\quad$ $c)\ y = x^2$

$d)\ y = x^2 + 2$ $\quad$ $e)\ y = x^2 + 2x$ $\quad$ $f)\ y = 2x^2 - 6x + 1$

1. *Continued …Use the* ***DEFINITION OF THE DERIVATIVE*** *to find* $\frac{dy}{dx}$.

$g)\ y = \frac{1}{x}$ $h)\ y = \frac{1}{x+2}$ $i)\ y = \frac{1}{3x}$

$j)\ y = \sqrt{x}$ $k)\ y = \sqrt{x+4}$ $l)\ y = \sqrt{2x}$

3. 4 - Fundamental Limit Involving Trig.

$$\boxed{\lim_{h \to 0} \frac{\sin h}{h} = 1}$$

Examples: Evaluate the following.

$a)\quad \lim_{h \to 0} \frac{\sin 6h}{h}$

$b)\quad \lim_{h \to 0} \frac{2\sin 5h}{3h}$

You Try: Evaluate the following.

$$\lim_{h \to 0} \frac{\sin 5h}{5h}$$

Fundamental Limit Involving e

Theorem 4.67. The Fundamental Limit of Calculus.

$$e = \lim_{h \to 0}(1+h)^{1/h} \qquad \text{and} \qquad e = \lim_{n \to \infty}\left(1+\frac{1}{n}\right)^n$$

Examples: Evaluate the following.

$a)$ $\lim_{x \to 0}(1+x)^{\frac{5}{x}}$

$b)$ $\lim_{x \to \infty}\left(1+\frac{5}{x}\right)^x$

You Try: Evaluate the following.

$a)$ $\lim_{x \to 0}(1+2x)^{\frac{1}{x}}$

$b)$ $\lim_{h \to \infty}\left(1+\frac{1}{3h}\right)^h$

3.4 - PRACTICE QUESTIONS

1. *Use the fundamental trigonometric limit to evaluate each of the following:*

$a)\ \lim_{h\to 0} \frac{\sin 3h}{h}$ $\qquad b)\ \lim_{h\to 0} \frac{\sin 5h}{h}$ $\qquad c)\ \lim_{h\to 0} \frac{\sin 6h}{6h}$

$d)\ \lim_{h\to 0} \frac{4\sin h}{h}$ $\qquad e)\ \lim_{h\to 0} \frac{3\sin 5h}{2h}$ $\qquad f)\ \lim_{h\to 0} \frac{3\sin 3h}{5h}$

$g)\ \lim_{n\to 0} \frac{\sin 7n}{\sin 2n}$ $\qquad h)\ \lim_{x\to 0} \frac{2\sin 4x}{\sin 3x}$

2. *Use the fundamental limit involving e to evaluate each of the following:*

$a)\ \lim_{h\to 0} \left(1+h\right)^{\frac{3}{h}}$ $\qquad b)\ \lim_{x\to 0} \left(1+\frac{x}{5}\right)^{\frac{1}{x}}$ $\qquad c)\ \lim_{u\to 0} \left(1+3u\right)^{\frac{1}{u}}$

$d)\ \lim_{y\to 0} \left(1+4y\right)^{\frac{2}{y}}$ $\qquad e)\ \lim_{t\to 0} \left(1+\sin t\right)^{\frac{1}{\sin t}}$ $\qquad f)\ \lim_{h\to 0} \left(1+2h\right)^{\frac{1}{h}}$

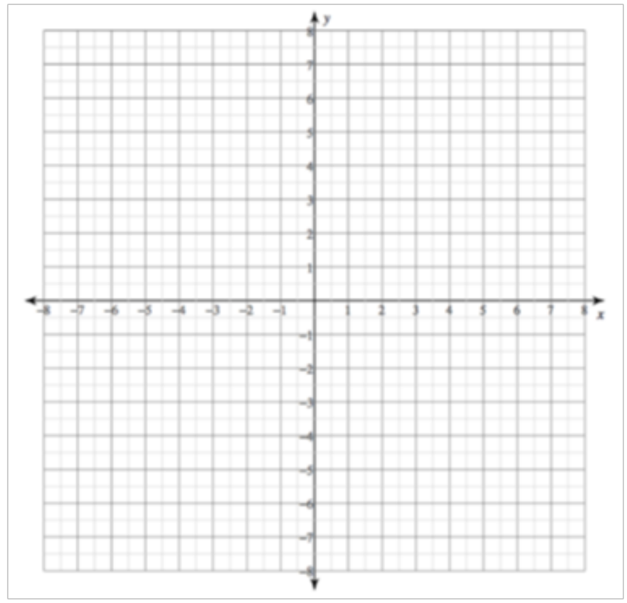
3. *Use the fundamental limit involving e to evaluate each of the following:*

$a)\ \lim_{n\to +\infty} \left(1+\frac{2}{n}\right)^{n}$ $\qquad b)\ \lim_{n\to +\infty} \left(1+\frac{3}{n}\right)^{n}$ $\qquad c)\ \lim_{m\to +\infty} \left(1+\frac{1}{m^2}\right)^{m^2}$

3.5 - Piece-wise Limit Functions

Example: Use the piecewise functions to find the given values. Sketch below to help.

0. $f(x) = \begin{cases} \frac{1}{x+6}, & x < -2 \\ 2^x, & -2 \le x < 0 \\ x^2 - 4, & x \ge 0 \end{cases}$



a. $\lim_{x \to -2} f(x) =$

b. $\lim_{x \to -2^-} f(x) =$

c. $\lim_{x \to -2^+} f(x) =$

d. $\lim_{x \to 0} f(x) =$

e. $\lim_{x \to 0^-} f(x) =$

f. $\lim_{x \to 0^+} f(x) =$

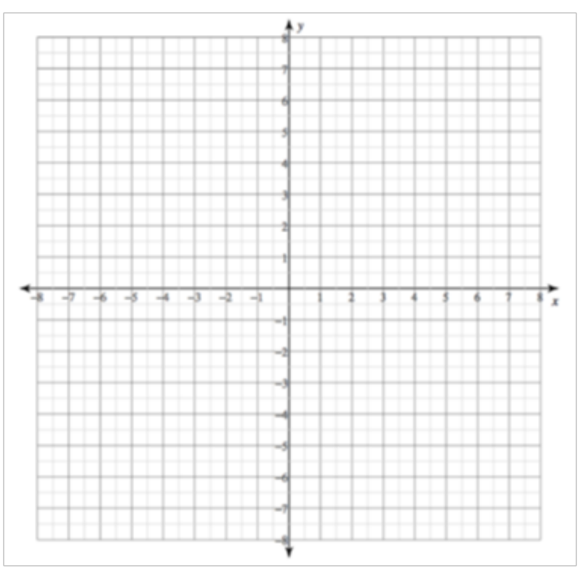
g. $f(-2) =$

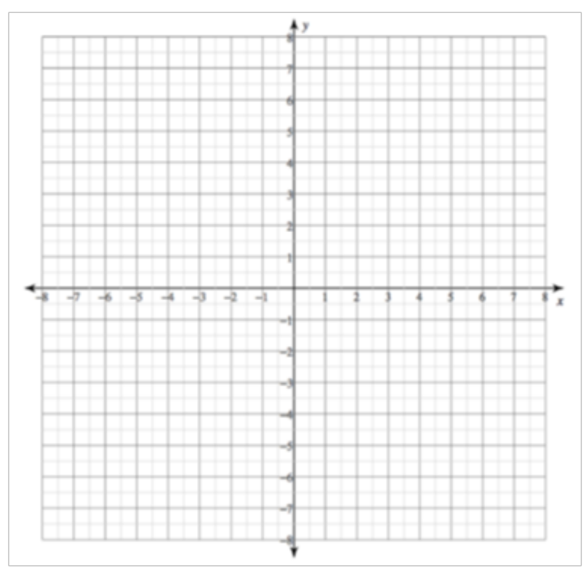
h. $f(0) =$

Example:
Create your own piece-wise, non-continuous, function limit question with the following criteria:

a) > parabola opening up with 2 terms
> oblique line with negative slope

b) > exponential
> parabola opening down with 3 terms





3.5 - PRACTICE QUESTIONS

Use the piecewise functions to find the given values. Sketch below to help.

1.

$$g(x) = \begin{cases} \sqrt{5-x}, & x < -4 \\ x^2 - 5, & -4 \le x < 2 \\ x - 3, & x \ge 2 \end{cases}$$

a. $\lim\limits_{x \to 2^-} g(x) =$

b. $\lim\limits_{x \to -4^+} g(x) =$

c. $g(2) =$

d. $\lim\limits_{x \to -4^-} g(x) =$

e. $\lim\limits_{x \to 2^+} g(x) =$

f. $\lim\limits_{x \to 2} g(x) =$

g. $\lim\limits_{x \to -4} g(x) =$

h. $g(-4) =$

2.

$$w(\theta) = \begin{cases} \sin\theta, & \theta \le \pi \\ \cos\theta, & \pi < \theta < 2\pi \\ \tan\theta, & \theta > 2\pi \end{cases}$$

a. $\lim\limits_{x \to \pi^-} w(\theta) =$

b. $w(\pi) =$

c. $\lim\limits_{x \to \pi^+} w(\theta) =$

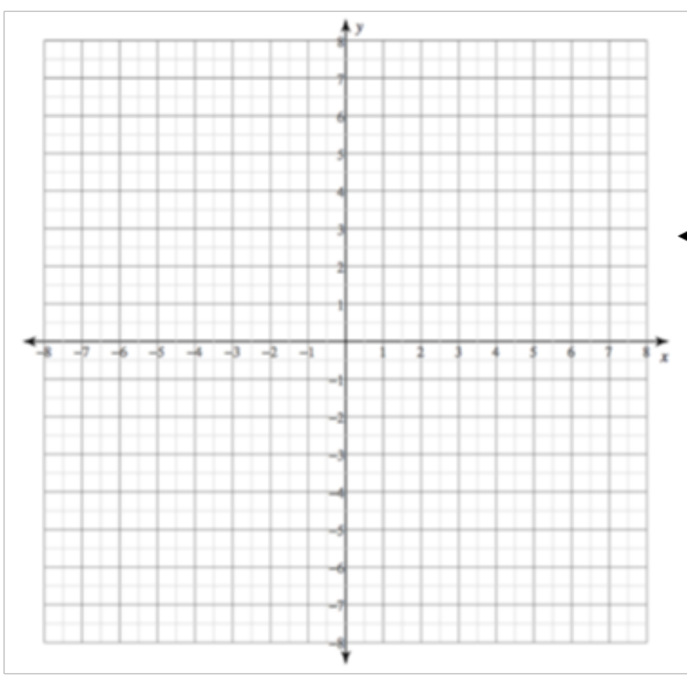
d. $\lim\limits_{x \to 2\pi^-} w(\theta) =$

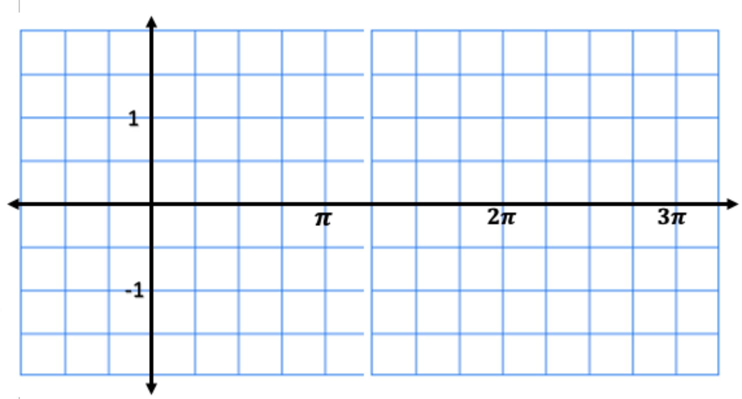
e. $\lim\limits_{x \to \pi} w(\theta) =$

f. $\lim\limits_{x \to 2\pi^+} w(\theta) =$

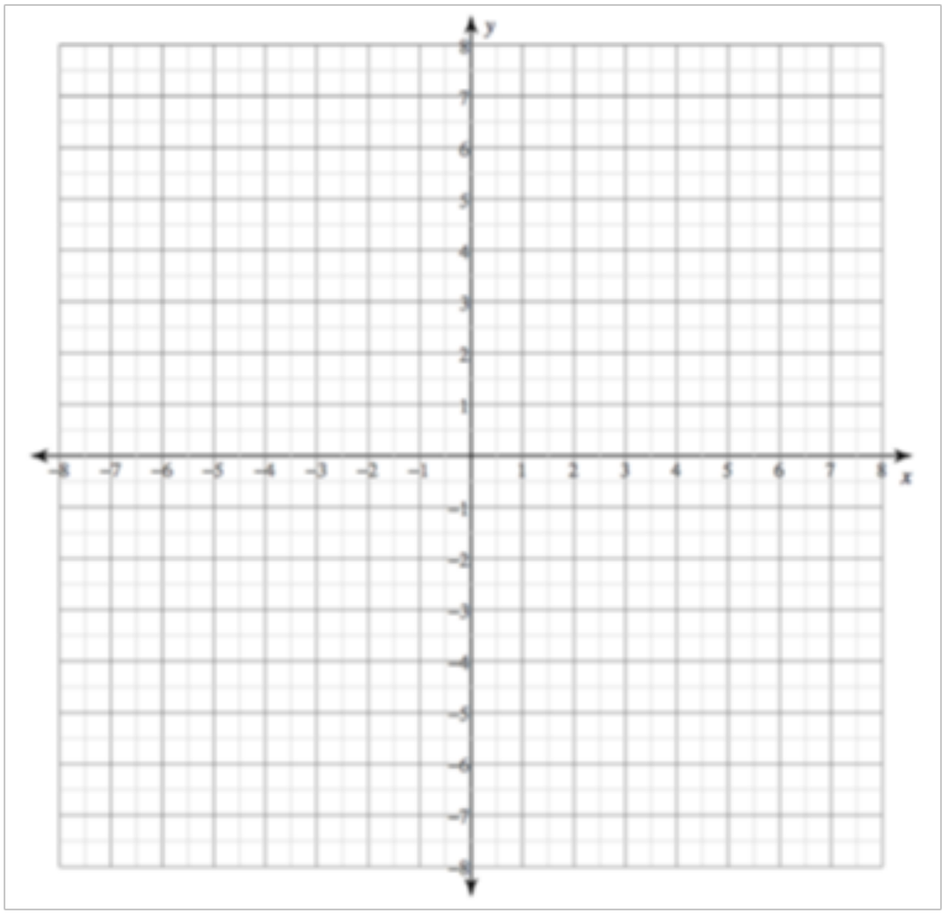
g. $\lim\limits_{x \to 2\pi} w(\theta) =$

h. $w(2\pi) =$

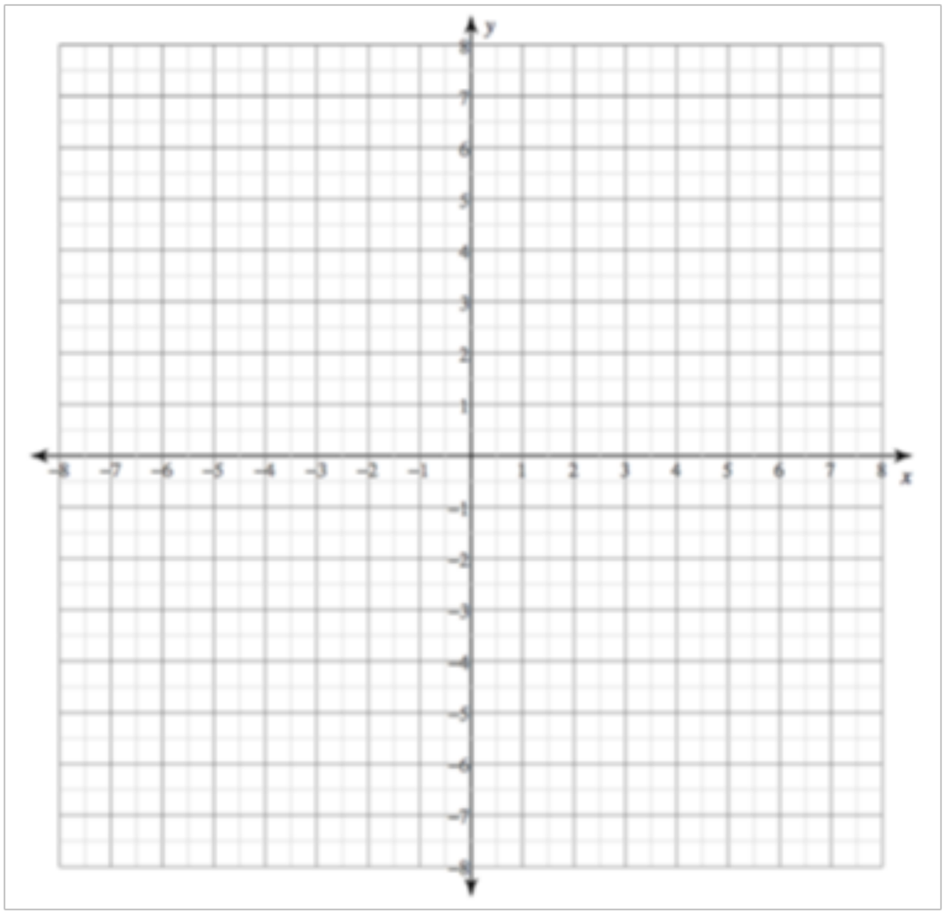


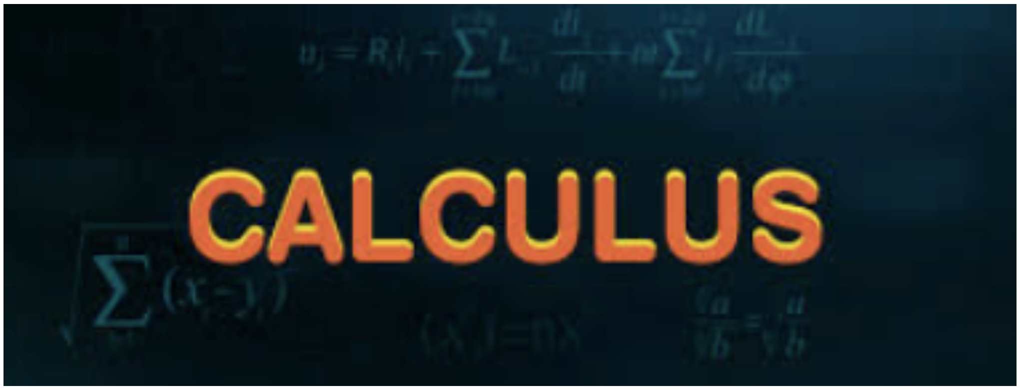


3. *Create your own piece-wise, non-continuous limit question with the following criteria:*
   > oblique line (negative slope)
   > rational function (VA: x=2)



4. *Create your own piece-wise, non-continuous limit question with the following criteria:*
   > parabola
   > radical function





SEMINAR NOTES

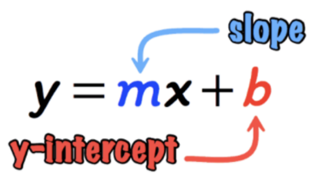
UNIT 4

4.1 – Equations of Lines

Finding Slope

Slope-Intercept Form of a Line ($y = mx + b$)

The slope-intercept is the most "popular" form of a straight line. Many students find this useful because of its simplicity. One can easily describe the characteristics of the straight line even without seeing its graph because the **slope** and *y*-**intercept** can easily be identified or read off from this form.



Finding *x* & *y* Intercepts

Intercepts:
An *x*-intercept is a point where a function crosses the *x*-axis. The *y*-intercept is the point where a function crosses the *y*-axis.

Find the *x*-intercept: Let $y = 0$, because the ordinate of the point on the *x*-axis is 0. Then solve for *x*.

Find the *y*-intercept: Let $x = 0$, because the abscissa of the point on the *y*-axis is 0. Then solve for y.

Examples: Find the slope, *x* & *y* intercepts for the following.

a) $3x - 4y - 5 = 0$

b) $\frac{3x}{4} - \frac{y}{3} = 2$

Now, find the parallel (//) and perpendicular (⊥)slope of a & b above.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Writing an Equation of a Line $\boxed{y = mx + b}$

Example: Write an equation given the two points (2, -3) and (0, 6).

Example: Write an equation given a point (-5, 2) that is perpendicular to the line $-x - 4y = 3$.

Use the Derivative to find the Slope of a "Tangent Line" & a "Normal Line" to a Given Point

Example:

$a)\ y = 5x^4 - 7x^3\ \ at\ \ (1, -2)$

$b)\ \ y = \frac{x-3}{2x^2+3}\ \ at\ \ (3, 0)$

You Try:

$a)\ y = 6x^2 - 2x\ \ at\ \ (-1, 8)$

$b)\ y = (3x - 1)^2\ \ at\ \ (1, 4)$

Tangent & Normal Line Equations

We will learn how to find the slope and equation of the tangent and normal to a curve at a given point using derivatives.

The derivative of a curve at a point tells us the slope of the tangent line to the curve at that point and there are many different techniques for finding the derivatives of different functions. We can utilize these differentiation techniques to help us find the equation of tangent lines to various differentiable functions.

First, let us recall exactly what we mean by the tangent to a curve a point.

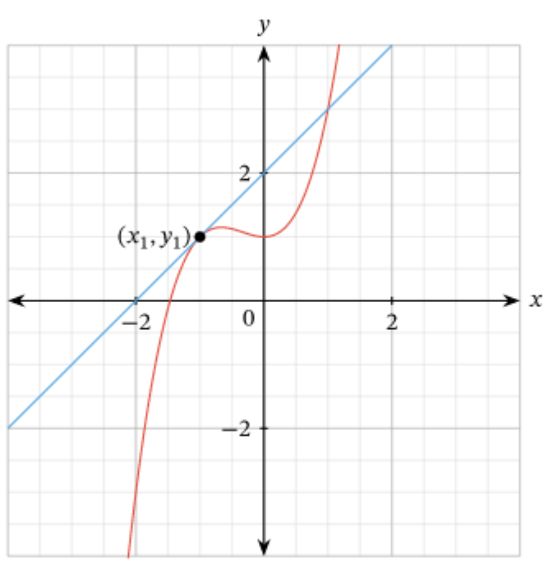
Definition: The Tangent Line to a Curve at a Point

For a curve $y = f(x)$ and point (x_1, y_1) on the curve, we say that the line $ax + by + c = 0$ is the tangent line to the curve at the point (x_1, y_1) if

- the tangent line passes through the point (x_1, y_1);
- the curve and tangent line have the same slope at the point (x_1, y_1).

I'll refer the line equation as:
$y = mx + b$

In the definition above, we state that our tangent line and curve will have the same slope at the point (x_1,y_1). This means that, around the point (x_1,y_1), the line will only touch the curve.



So far, we have been focused on tangent lines. However, there is another important type of line we need to consider called a **normal line**. A normal line to a curve at a point is very similar to the tangent line; the only difference is that the normal line will be perpendicular to the tangent line.

Definition: The Normal Line to a Curve at a Point

For a curve $y = f(x)$ and point (x_1, y_1) on the curve, we say that the line $ax + by + c = 0$ is the normal line to the curve at the point (x_1, y_1) if

- the point (x_1, y_1) lies on our line;
- this line is perpendicular to the tangent line of our curve at this point.

I'll refer the line equation as:
$y = mx + b$

Find the Equation in Slope-Intercept Form of a "Tangent Line" & Normal Line to a Given Point

Example:

$a)\ y = 5x^4 - 7x^3\ \ at\ \ (1, -2)$

$b)\ y = \dfrac{x-3}{2x^2+3}\ \ at\ \ (3, 0)$

You Try:

$a)\ y = 6x^2 - 2x\ \ at\ \ (-1, 8)$

$b)\ y = (3x-1)^2\ \ at\ \ (1, 4)$

4.1 - PRACTICE QUESTIONS

1. *Find the slope, x-intercept and y-intercept of each of the following lines:*

$a)\ 2x-4y+8=0$ $\qquad b)\ \frac{2}{3}x-\frac{1}{4}y=2$ $\qquad c)\ \frac{x}{2}-\frac{y}{5}=4$

2. *Find the slope of a line parallel to and the slope of a line perpendicular to each of the lines in question #1.*

3. *Find the equation in slope-intercept form of the line passing through:*

$a)\ (-1,2)$ *with slope of* $-\frac{1}{2}$ $\qquad c)\ (2,-1)$ *and parallel to* $3x-2y=-6$

$b)$ *the pts* $(3,1)$ *and* $(-2,-5)$ $\qquad d)\ (2,-1)$ *and perpendicular to* $3x-2y=-6$

4. *Use algebra and geometry to find the equation in slope-intercept form of the tangent line*
to the given circle at the given point.

$a)\ x^2+y^2=25$ *at* $(-3,4)$ $\qquad b)\ x^2+y^2=169$ *at* $(-5,-12)$

5. *Use the first derivative to find the slope of the tangent line to the given curve at the given*
point:

$a)\ \ y = 2x^2 + 6 \quad at \ \ (-1, 8)$

$b)\ \ y = -x^2 + 2x - 3 \quad at \ \ (2, 3)$

$c)\ \ y = 4 - 3x^3 \quad at \ \ (1, 1)$

$d)\ \ y = \dfrac{3x-1}{x+3} \quad at \ \ (-2, -7)$

6. *Find the slope of the normal line to the given curve at the given point for each of the curves in question #5.*

a) b) c) d)

7. *Find the equation in slope-intercept form of the tangent line to the given curve at the given point for each of the curves in question #5.*

a) b)

c) d)

8. *Find the equation in slope-intercept form of the normal line to the given curve at the given point for each of the curves in question #5.*

a) b)

c) d)

9. For each curve below find the equation of the (i) tangent line and (ii) normal line to the given curve at the given point:

$a)\ \ y=(x^3-5x+2)(3x^2-2x)\quad at\ \ (1,-2)$ $\qquad b)\ \ y=\sqrt{16x^3}\quad at\ \ (4,32)$

10. Tangent lines are drawn to the parabola $y=x^2\ \ at\ (2,4)\ and\ \left(\dfrac{-1}{8},\dfrac{1}{64}\right).$

Prove that the tangents are perpendicular.

11. *Find a point on the parabola* $y = -x^2 + 3x + 4$ *where the slope of the tangent line is 5.*

12. *Find the equation of the normal line to the curve* $y = -x^2 + 5x$ *that has slope of -2.*

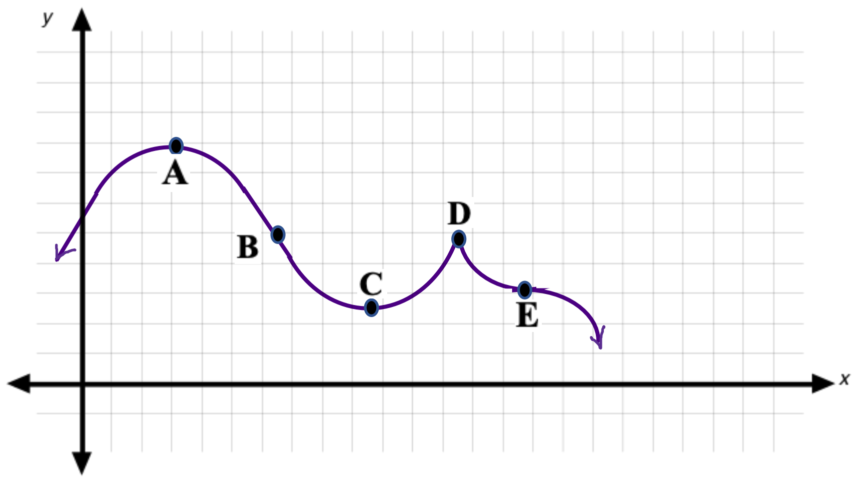
13. *Find the equations of the tangent lines to the curve* $y = 2x^2 + 3$ *that pass through the point (2, -7).*

14. *Prove the curve* $y = -2x^3 + x - 4$ *has* <u>*no*</u> *tangent with a slope of 2.*

15. *At what points on the curve* $y^3 - 3x = 5$ *is the slope of the tangent line equal to 1?*

***BE SURE TO DO MULTIPLE CHOICE QUESTIONS ONLINE **VERSION 7**
#56 - 58, 60, 66, 69 – 71, 128, 130
https://moodle.sd79.bc.ca/pluginfile.php/1871/mod_resource/content/5/AB%20Calculus%20Version%207.pdf

4.2 - What's the Point

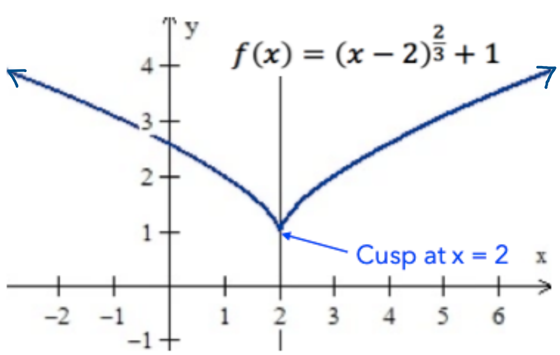


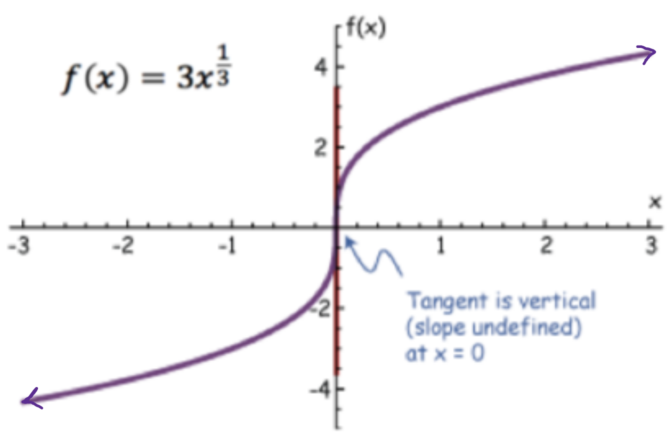
A ___________________
B ___________________
C ___________________
D ___________________
E ___________________

Relative vs. Absolute

- The term 'extrema' refers to maximums and/or minimums.
- The general term for maximums or minimums is 'extremum'.
- Extrema can be relative or absolute.
- An absolute minimum/maximum is the greatest/least value that a function assumes over its domain.
- A relative max/min may not be the greatest/least over its domain, but it is the greatest/least over some interval in the domain.
- Extrema are always values of the function; they are the y-coordinates of each max or min.

Functions that are continuous over its domain $(-\infty, \infty)$, but non-differentiable.



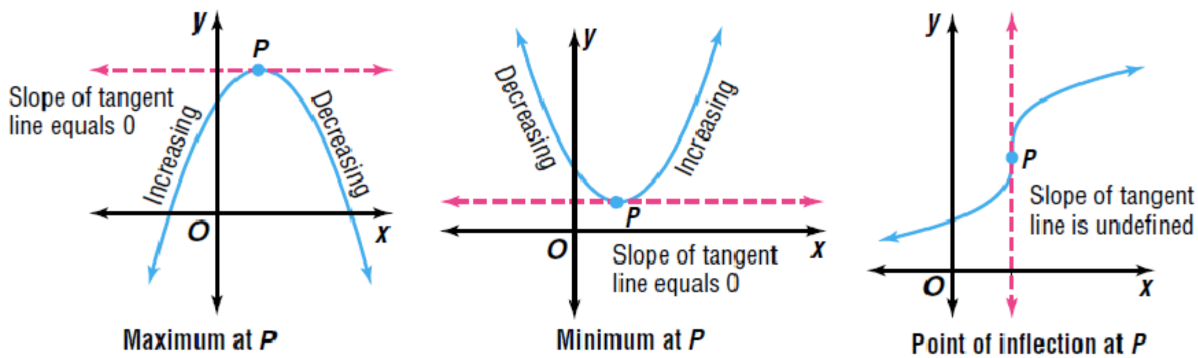


Critical Points

Definition and Types of Critical Points

• Critical Points: those points on a graph at which a line drawn tangent to the curve is horizontal or vertical.

• Polynomial equations have three types of critical points- maximums, minimum, and points of inflection.



HIGH POINT | LOW POINT | CHANGE IN CURVATURE

In words, to find Critical Points you first _________________ of the function, then set it equal ______ and ___________. To find the y value you plug ____ into ______________ function and solve.

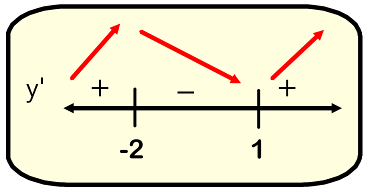
Note: extrema is just max and min, NOT points of inflection.

Example: Use Calculus to find the Critical Points

a) $y = 2x^2 + 8x - 2$

b) $y = 2x^3 - 6x^2$

Use Calculus showing a <u>THUMBNAIL SKETCH</u> to find <u>TURNING POINTS</u> of the following function:

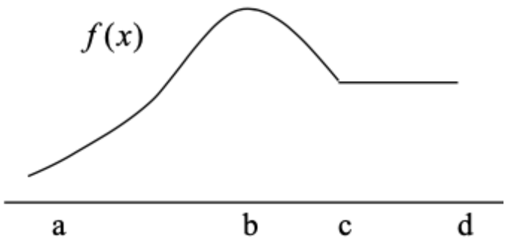


A turning point may be either a relative maximum or a relative minimum (also known as local minimum and maximum). If the function is differentiable, then a turning point is a stationary point; however not all stationary points are turning points.

Example: $y = 2x^3 + 9x^2 + 12x$

You Try: $y = \frac{1}{3}x^3 + \frac{5}{2}x^2 - 6x + 1$

Increasing and Decreasing Test



If $f'(x) > 0$ on an interval, then f is ______________ on that interval. This means that the graph goes _______________________________. Eg.

If $f'(x) < 0$ on an interval, then f is ______________ on that interval. This means that the graph goes _______________________________. Eg.

If $f'(x) = 0$ on an interval, then f is ______________ on that interval. This means that the graph is __________. Eg.

Steps to finding the intervals on which a function is increasing or decreasing

1) Set up a **sign chart** for $f'(x)$ using the critical points. The **critical numbers** occur where $f'(x)$ ______ or $f'(x)$ ___________________. These are break points in which the graph could possibly change directions.

2) On each interval determined by the critical points, pick a point at random and plug it into $f'(x)$.

3) If a point gives a positive value for $f'(x)$, then you know that $f'(x)$ is positive on the interval, and hence that the function ______________. Put a "+" above the interval and draw an upward-sloping line below it.

4) Likewise, if a point gives a negative value for $f'(x)$, then you know that $f'(x)$ is negative on the interval, and hence that the function ______________. Put a "-" above the interval and draw a downward-sloping line below it.

Examples: Find the intervals on which the functions are increasing and decreasing.

a) $f(x) = x^2 + 2x + 2$

b) $f(x) = 2x^3 - 3x^2 - 12x$

First Derivative Test

The use of $f'(x)$ to classify a critical point as a max or a min is often called the first derivative test.

First Derivative Test Suppose that c is a critical number of a continuous function f.

a) if f' changes from positive to negative at c, then f has a ______________________ at c.

b) if f' changes from negative to positive at c, then f has a ______________________ at c.

c) if f' does not change sign at c, then f has no ________ _________________ or _________________ at c.

Example: Locate any extreme values of f. *(local maximum & minimum)

a) $f(x) = x^3 - 12x - 5$

b) $f(x) = x^4 - 2x^2 - 3$

Another useful method to find and classify all local extrema is the following:

SECOND DERIVATIVE TEST

Example:
Use Calculus to find and classify all Local extrema:

$$y = x^3 - 12x + 5$$

Second Derivative Test - Steps:

1^{st} $\Rightarrow$ find critical pts by determining where $f'(x) = 0$ or does not exist

2^{nd} $\Rightarrow$ find corresponding y values of critical pts

3^{rd} $\Rightarrow$ find $f''(x)$ to determine sign at C.P.

4^{th} $\Rightarrow$ If $f''(x) > 0 \therefore$ relative MIN. at C.P.
If $f''(x) < 0 \therefore$ relative MAX. at C.P.
If $f''(x) = 0 \therefore$ test fails

To summarize,

Second Derivative Test Suppose f'' is continuous near c.

a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a ___________ ________________ at c.

b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a ___________ ________________ at c.

c) If $f''(c) = 0$, the test ____________________. f may have a maximum, a minimum or neither. This test also fails when $f''(c)$ does not exist. Therefore, the First Derivative Test must be used.

Critical Points

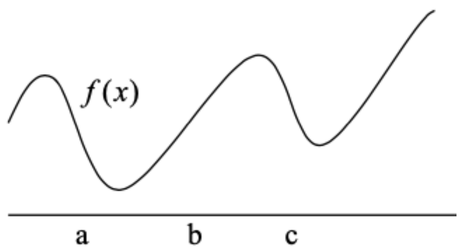
Stationary Points

Inflection Points

Turning Points

Concavity

If the graph of *f* lies *above* all of its tangents on an interval *I*, then it is called ______________ ___________________ on *I*. If the graph of *f* lies *below* all of its tangents on an interval *I*, then it is called _________________________________ on *I*.



If $f''(x) > 0$ for all x in *I*, then the graph of *f* is ___________ _______________ on *I*.
(Think concave up holds water.) eg.

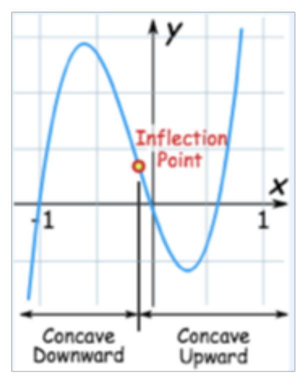
If $f''(x) < 0$ for all x in *I*, then the graph of *f* is ___________ _______________ on *I*.
(Think concave down spills water.) eg.

Definition - A point where the graph changes concavity is called an _______________________

INFLECTION POINTS

Example:
Use Calculus showing a to find **inflection point(s)** of the following function:

$$y = x^3 + 3x^2 + 3x$$



Now that you had found **inflection point(s)**…ask yourself, do they exist?
*Remember there must be a change in concavity!

We can use the **SECOND DERIVATIVE TEST** to find concavity change.

You Try:
Use Calculus showing a to find **inflection point(s)** of the following function:

$$y = x^4 - 4x^3 + 2$$

Review of what's going on

graph feature	f(x)	f'(x)	f"(x)	Notes
rising	slope > 0	+		
falling	slope < 0	−		
extrema: maximum	slope = 0	= 0 + on L, - on R	− at x_{max}	derivative may not exist at a max or min, e.g. **Cusps** or
extrema: minimum	slope = 0	= 0 - on L, + on R	+ at x_{min}	
inflection pt.	Curvature changes: ↔		= 0 potential inflection point	
concave up		− / +	+	
concave down		+ / −	−	

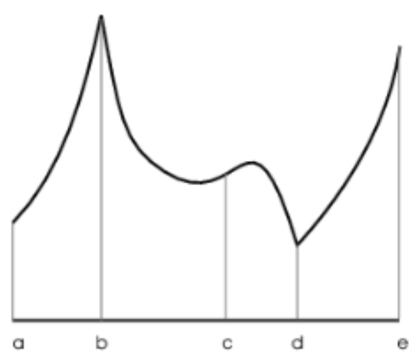
Steps to finding the intervals of concavity

1. Set up a **sign chart** for $f''(x)$ using the critical points. The **critical points** occur where $f''(x)$ ______ or $f''(x)$ __________________. These are break points in which the graph

   could possibly change concavity.

2. On each interval determined by the critical points, pick a point at random and plug it into $f''(x)$.
3. If a point gives a positive value for $f''(x)$, then you know that $f''(x)$ is positive on the interval, and hence that the function is ________________. Put a "+" above the interval and draw a ∪ below it.
4. Likewise, if a point gives a negative value for $f''(x)$, then you know that $f''(x)$ is negative on the interval, and hence that the function is __________________. Put a "-" above the interval and draw a ∩ below it.

Example – The graph of a function is pictured below. Determine the intervals on which the function is concave up and the intervals on which it is concave down. Find the x-coordinates of any inflection points.



Now it's time to put Increasing/Decreasing & Concavity together.

Example: Find the intervals where the following functions are:
a) increasing/decreasing
b) concave up/concave down

$f(x) = x^4 - 4x^3$

$f(x) = xe^x$

4.2 - PRACTICE QUESTIONS

1. *Use Calculus to find the* ***CRITICAL POINTS*** *of each of the following functions:*

$a)\quad y = x^2 - 6x + 5$ $\qquad b)\ y = 2x^3 - 24x$ $\qquad c)\ y = 2x^3 + 6x^2 + 6x$

2. *Use Calculus showing a* ***THUMBNAIL SKETCH*** *to find the* ***TURNING POINTS*** *of each. of the following functions*

$a)\ y = x^2 - 4x - 3$ $\qquad b)\ y = -2x^2 + 6x + 13$

$c)\ y = x^3 - 12x$ $\qquad d)\ y = 2x^3 + 9x^2 + 12x$

$e)\ y = x^3 + 3x^2 + 3x$ $\qquad f)\ y = 3x^4 - 4x^3$

3. *Find the local extrema (local maximum & minimum) using the FIRST DERIVATIVE TEST. Show a THUMBNAIL SKETCH.*

$a)\ \ y = 2x^3 + 3x^2 - 12x$ $\qquad$ $b)\ \ y = 4x^3 - 48x + 10$ $\qquad$ $c)\ \ y = \left(1 - x^2\right)^2 - 2$

4. *Use Calculus showing a* ***THUMBNAIL SKETCH*** *to find* ***the INFLECTION POINTS*** *of each of the following functions:*

$a)\ \ y = x^3 - 12x$ $\qquad$ $b)\ \ y = x^3 + 3x^2 + 3x$ $\qquad$ $c)\ \ y = x^2 - 4x - 3$

5. *Use the* ***SECOND DERIVATIVE TEST*** *to find and classify all local* ***extrema****:*

$a)\ \ y = x^2 - 10x + 3$ $\qquad$ $b)\ \ y = x^3 - 12x + 5$ $\qquad$ $c)\ \ y = x^4 - 2x^3$

6. *Use Calculus to find the turning points of each of the following polynomial functions (show a thumbnail sketch). Identify each turning points as a local maximum or a local minimum and sketch the graph near each turning point.*

$a)\ \ y = x^2 - 6x + 8$

$b)\ \ f(x) = x^3 + 2$

$c)\ \ g(x) = \frac{4}{3}x^3 - 4x$

$d)\ \ y = x^3 - 12x$

$e)\ \ y = -2x^5 + 1$

$f)\ \ h(x) = x^4 + 4x^3 - 8x^2$

***BE SURE TO DO MULTIPLE CHOICE QUESTIONS ONLINE **VERSION 7**
#23 - 37, 236 – 248, 272 - 292
https://moodle.sd79.bc.ca/pluginfile.php/1871/mod_resource/content/5/AB%20Calculus%20Version%207.pdf

EXTENDED QUESTIONS

PART I:

Find all intervals where the following functions are increasing or decreasing.

1. $y = -2x^3 + 3x^2 + 12x + 2$

2. $y = x^5 - 5x^3$

3. $y = \frac{x+2}{x-2}$

4. $y = \frac{x}{x^2-1}$

5. $y = x^2e^{-x^2}$

6. $y = xlnx$

7. $y = 3x^{\frac{2}{3}} - x^2$

8. $y = (2x - 4)^{\frac{1}{3}}$

PART II:

Find all intervals where the following functions are concave up or concave down.

1. $y = -2x^3 + 3x^2 + 12x + 2$

2. $y = x^5 - 5x^3$

3. $y = \frac{x+2}{x-2}$

4. $y = \frac{x}{x^2-1}$

5. $y = x^2e^{-x^2}$

6. $y = xlnx$

7. $y = 3x^{\frac{2}{3}} - x^2$

8. $y = (2x - 4)^{\frac{1}{3}}$

4.3 - Sketching Functions

The graph of a function is often a **useful way of visualizing the relationship of the function models** and manipulating a mathematical expression for a function can throw light on the function's properties. Functions presented as expressions can model many important phenomena.

Goal: Use first and second derivatives to make a rough sketch of the graph of a function $f(x)$.

Recall:

Critical points: Points c in the domain of $f(x)$ where $f'(c)$ does not exist or $f'(c) = 0$.

Monotony: AKA "Thumbnail Sketch"

$f'(x) > 0 \Rightarrow f(x)$ increasing.
$f'(x) < 0 \Rightarrow f(x)$ decreasing.

Local extrema: Appear at points c in the domain of $f(x)$ where $f(x)$ changes from increasing to decreasing ($f(c)$ maximum) or from decreasing to increasing ($f(c)$ minimum).

First derivative test:

Let $f'(c) = 0$. Then:

$f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c \Rightarrow f(c)$ is local max.
$f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c \Rightarrow f(c)$ is local min.

Concavity:

$f''(x) > 0 \Rightarrow f(x)$ concave up.
$f''(x) < 0 \Rightarrow f(x)$ concave down.

Inflection points: Points $(x, f(x))$ where the graph of $f(x)$ changes its concavity.

Inflection point test:

Let $f''(c) = 0$.
If $f''(x)$ changes its sign at $x = c$ then $f(x)$ has a inflection point at $x = c$.

Second derivative test:

Let $f'(c) = 0$. Then:

$f''(c) < 0 \Rightarrow f(c)$ is a local maximum
$f''(c) > 0 \Rightarrow f(c)$ is a local minimum

Critical points: Points where $f'(x)$ or $f''(x)$ has a sign change.

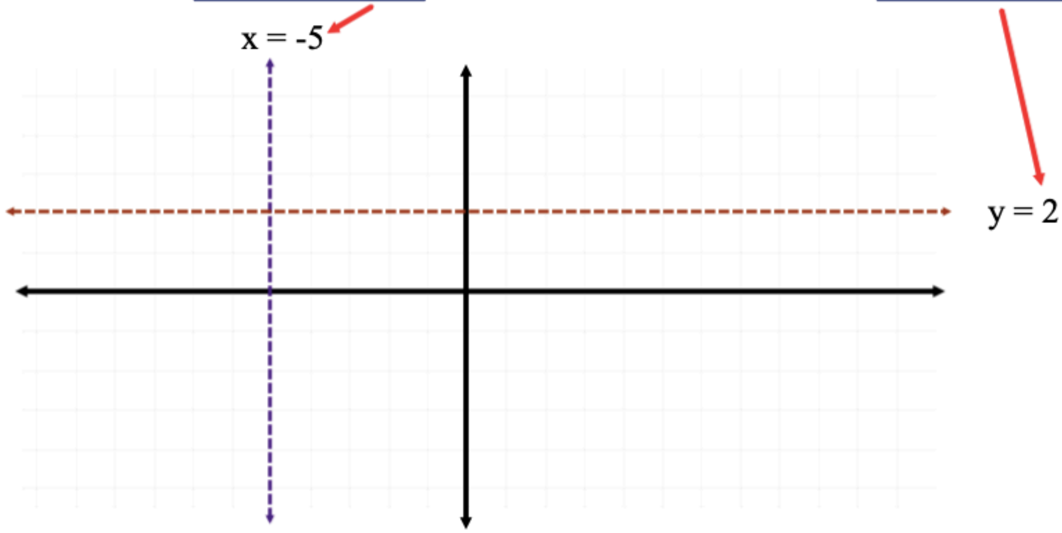
Those are the points where the graph of $f(x)$ may changes its features. We will concentrate to find those points

Asymptotes

There are:

Vertical Asymptotes, which are derived from the denominator.

Horizontal Asymptotes, which are derived from the numerator.

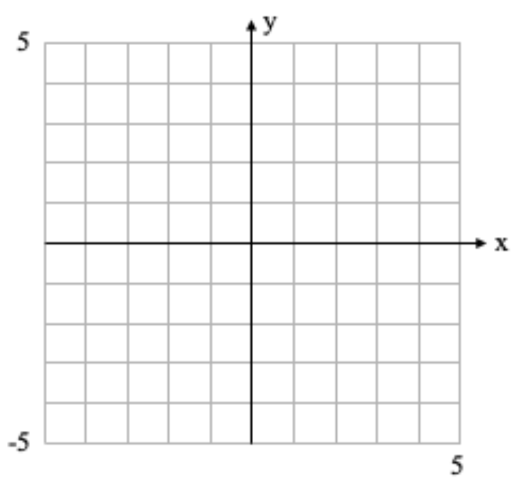


Vetical Asymptotes

Look at This Example:

Use Algebra to find the vertical asymptote(s), if they exist. Sketch the graph near the asymptote(s).

$$y = \frac{1}{x+2}$$



Horizontal Asymptotes

DON'T FORGET
Slant Asymptotes
deg N(x) 1 > deg D(x)

A "recipe" for finding a horizontal asymptote of a rational function:

Let

deg N(x) = the degree of a numerator and **deg D(x)** = the degree of a denominator.

deg N(x) = deg D(x)	deg N(x) < deg D(x)	deg N(x) > deg D(x)
$y = \dfrac{\text{leading coefficient of N(x)}}{\text{leading coefficient of D(x)}}$	$y = 0$ which is the x-axis	There is **no** horizontal asymptote.

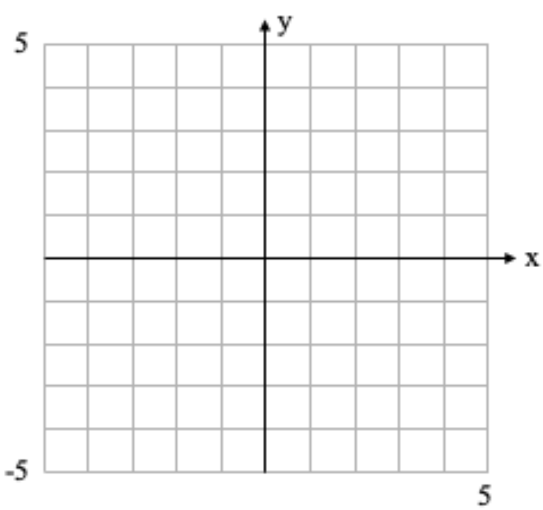
Another way of finding a horizontal asymptote of a rational function:

Divide every term by the highest power in the denominator.
Then plug in 100 or -100 into all x values.

Look at This Example:

Use Algebra to find the horizontal asymptote(s), if they exist.
Sketch the graph near the asymptote(s).

$$m(x) = \frac{3x^3 - 2}{x^3 + 1}$$



Oblique or Slant Asymptotes

When the degree of the numerator of a rational function exceeds the degree of the denominator by one then the function has **oblique asymptotes**. To find these asymptotes, you need to use polynomial long division and the non-remainder portion of the function becomes the oblique asymptote.

Examples:

$$f(x) = \frac{x^2-3x-4}{x-2}$$

Slant Asymptote:

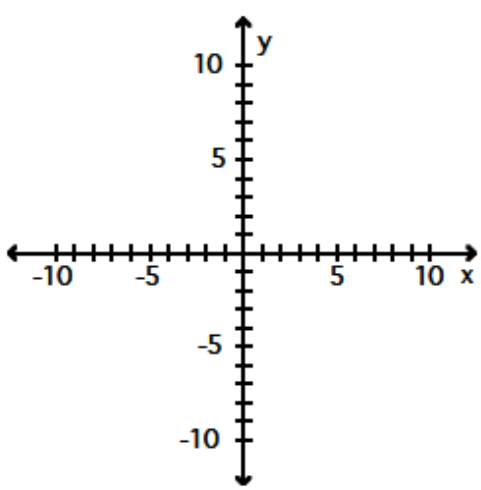
You Try:

$$f(x) = \frac{x^2-1}{2x-4}$$

Slant Asymptote:

Graph the function.

5) $f(x) = \frac{x^2+4}{x}$



Graph Sketching

Main Steps:

1. Determine the domain looking for points of discontinuity.

2. Find asymptotes and intercepts

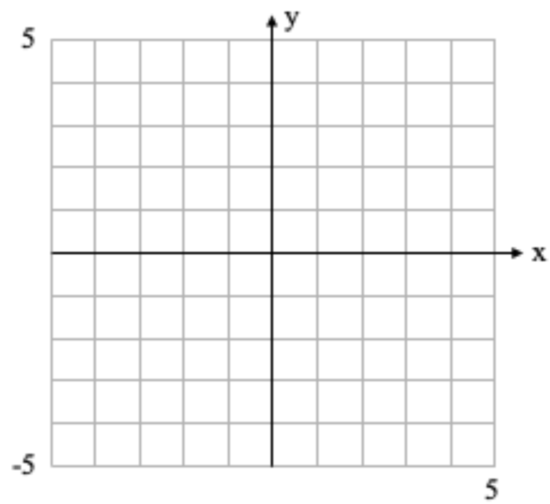
3. Find points with $f'(x) = 0$ and mark sign of $f'(x)$ on number line.

4. Find points with $f''(x) = 0$ and mark sign of $f''(x)$ on number line.

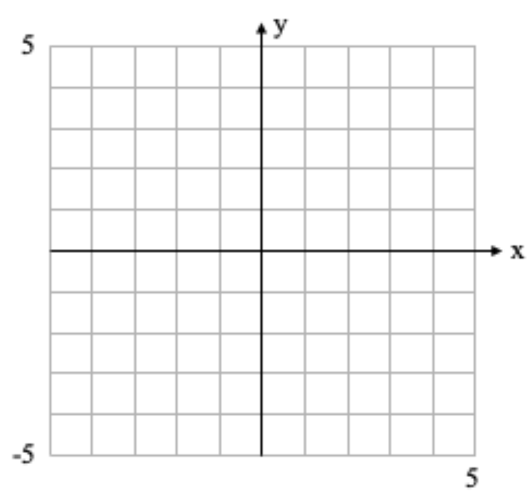
5. Sketch graph.

For each example, find intercepts, C.P., intervals where function is increasing & decreeing, I.P., intervals where function is CU, CD, and extrema(s).

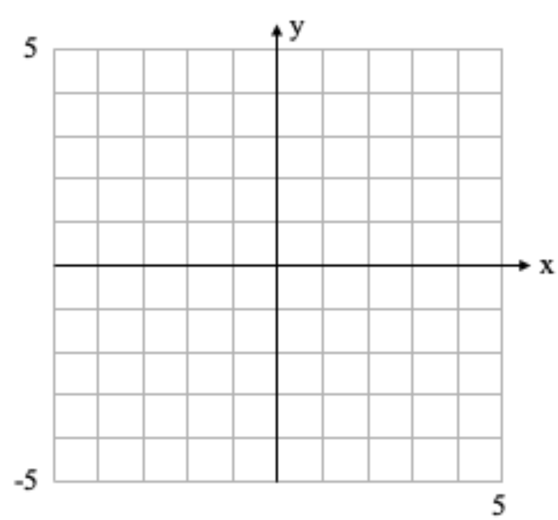
Example: Sketch the graph of $f(x) = 3x^4 + 5x^3 - 3x - 5$



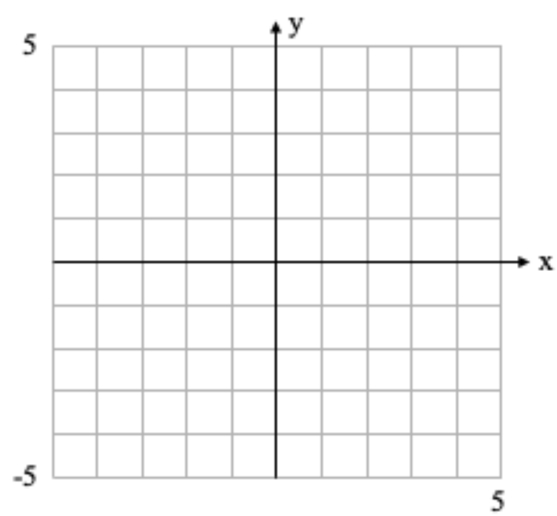
Example: Sketch the graph of $f(x) = \dfrac{x}{x^2 - 9}$.



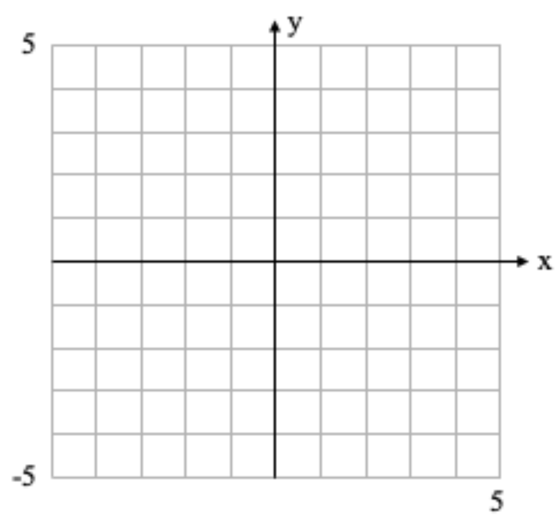
Example: Sketch the graph of $f(x) = x(x-2)^{2/3}$.



Example: Sketch the graph of $y = x^2 ln(2x)$



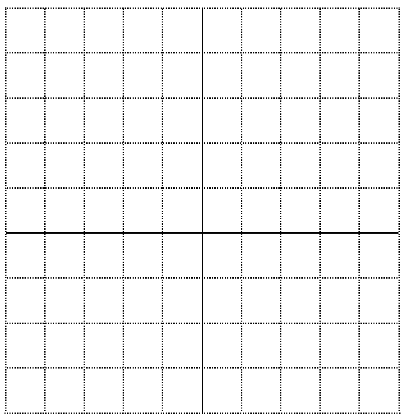
Example: Sketch the graph of $y = sin^2(x)$.



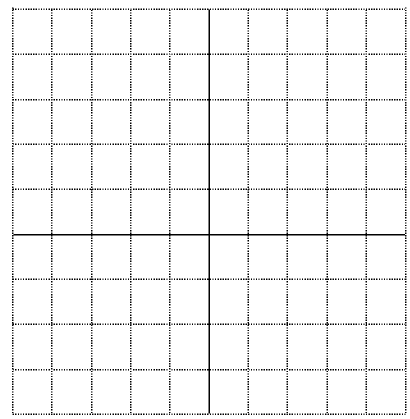
4.3 - PRACTICE QUESTIONS

1. *Use algebra to find* ***vertical asymptotes****, if they exist, of each function and sketch the graph near the asymptotes.*

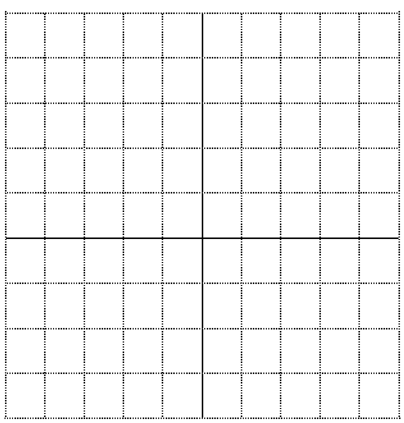
$a)\ \ y = \dfrac{1}{x+2}$



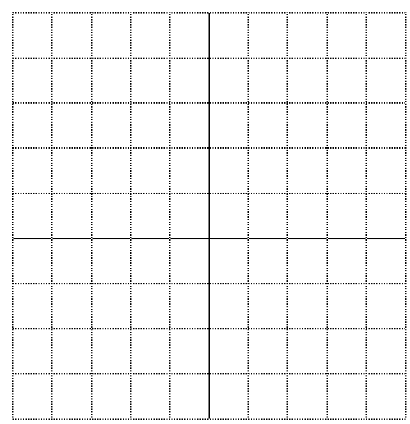
$b)\ \ f(x) = \dfrac{5}{(x-1)(x+3)}$



$c)\ \ g(x) = 6x^3 - 5x - 3$

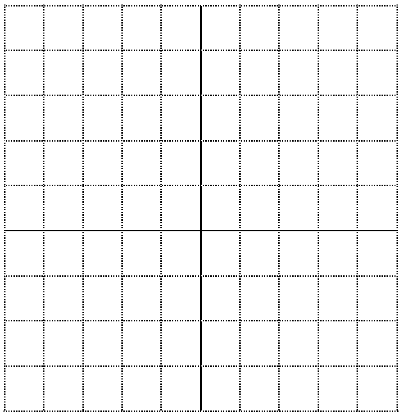


$d)\ \ h(x) = \dfrac{-6x}{x^2 - 4}$

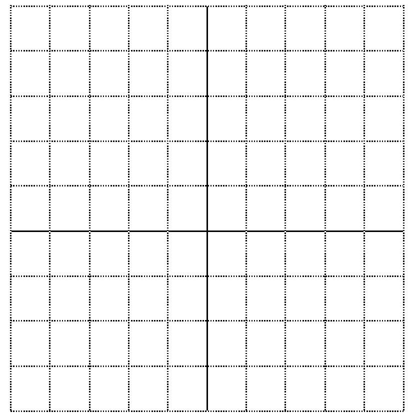


2. *Use Calculus to find the* ***horizontal asymptotes****, if they exist, of each function and sketch the graph near the asymptotes.*

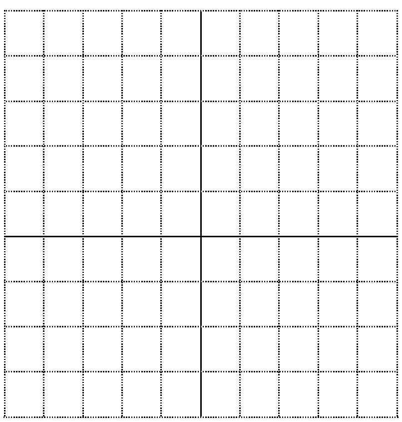
$a)\ \ y = \dfrac{-2}{x}$



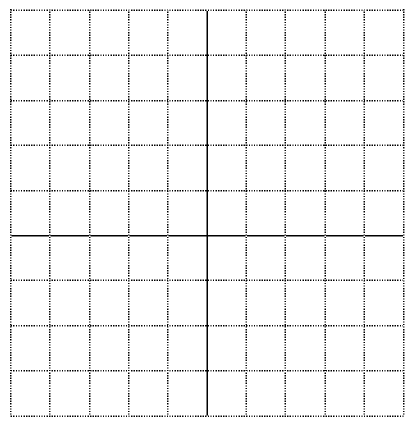
$b)\ \ f(x) = \dfrac{-2}{x+1}$



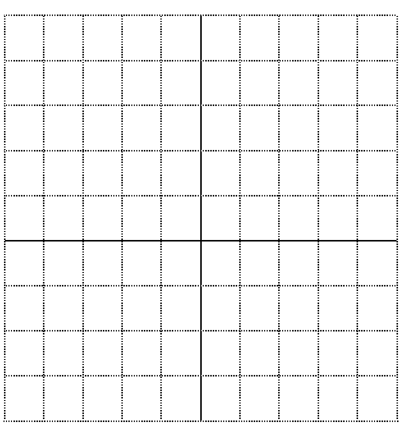
$c)\ \ g(x) = \dfrac{5}{x+1}$



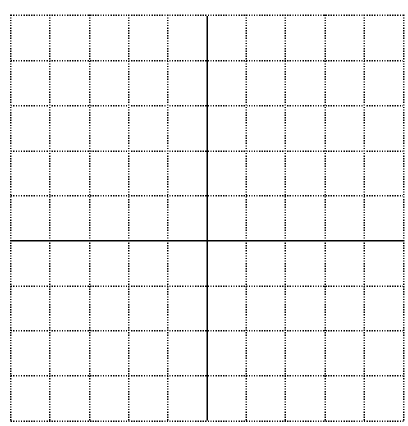
$d)\ \ h(x) = 4x^3 - 2x^2 + 5$



$e)\ \ y = \dfrac{3x^3 - 2}{x^3 + 1}$



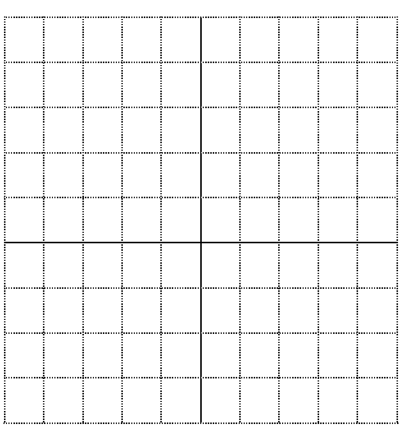
$f)\ \ k(x) = \dfrac{7x^3}{(2x-1)^3}$

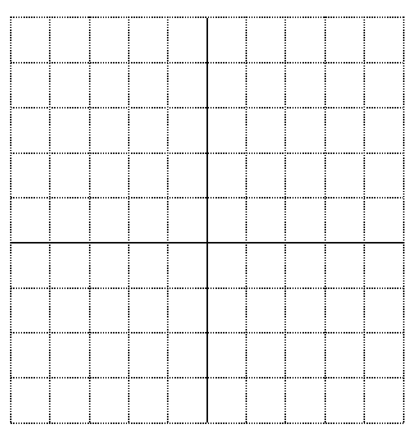


3. *Make a rough sketch of each function below. Indicate* ***x-intercept****,* ***y-intercept****,* ***turning points****,* ***vertical*** *and* ***horizontal asymptotes****.*

$a)\ y = \dfrac{x+4}{x+2}$

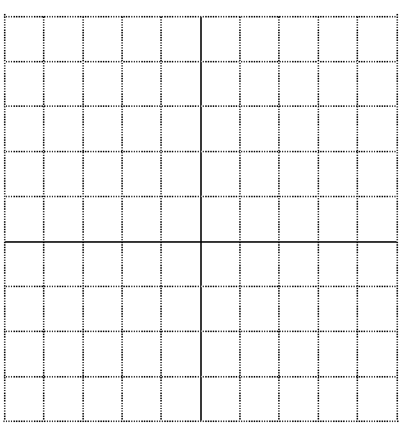
$b)\ f(x) = \dfrac{1}{1+x^2}$

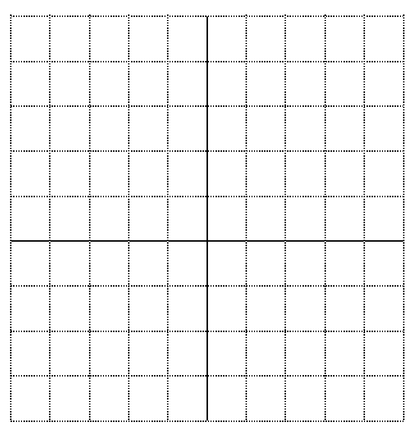




$c)\ \ g(x) = \dfrac{1}{x} - \dfrac{1}{x^2}$

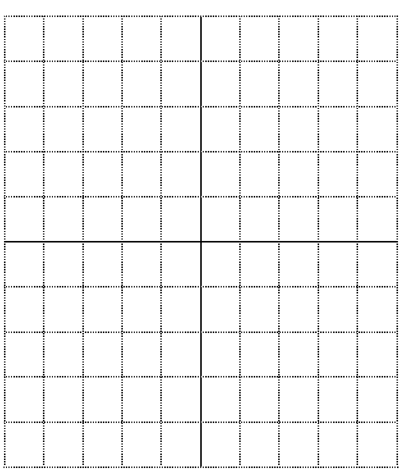
$d)\ \ h(x) = 2x^3 - 2x^2 - 5x + 5$

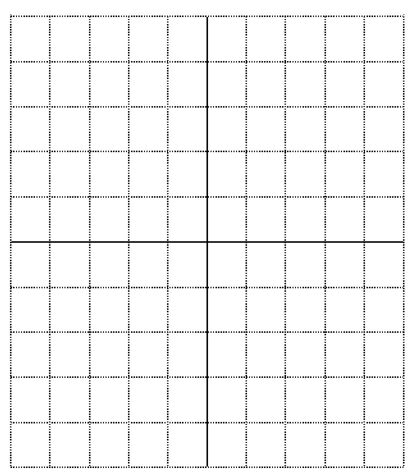




$e)\ \ y = \dfrac{x+1}{x}$

$f)\ \ y = x^2 + \dfrac{8}{x}$

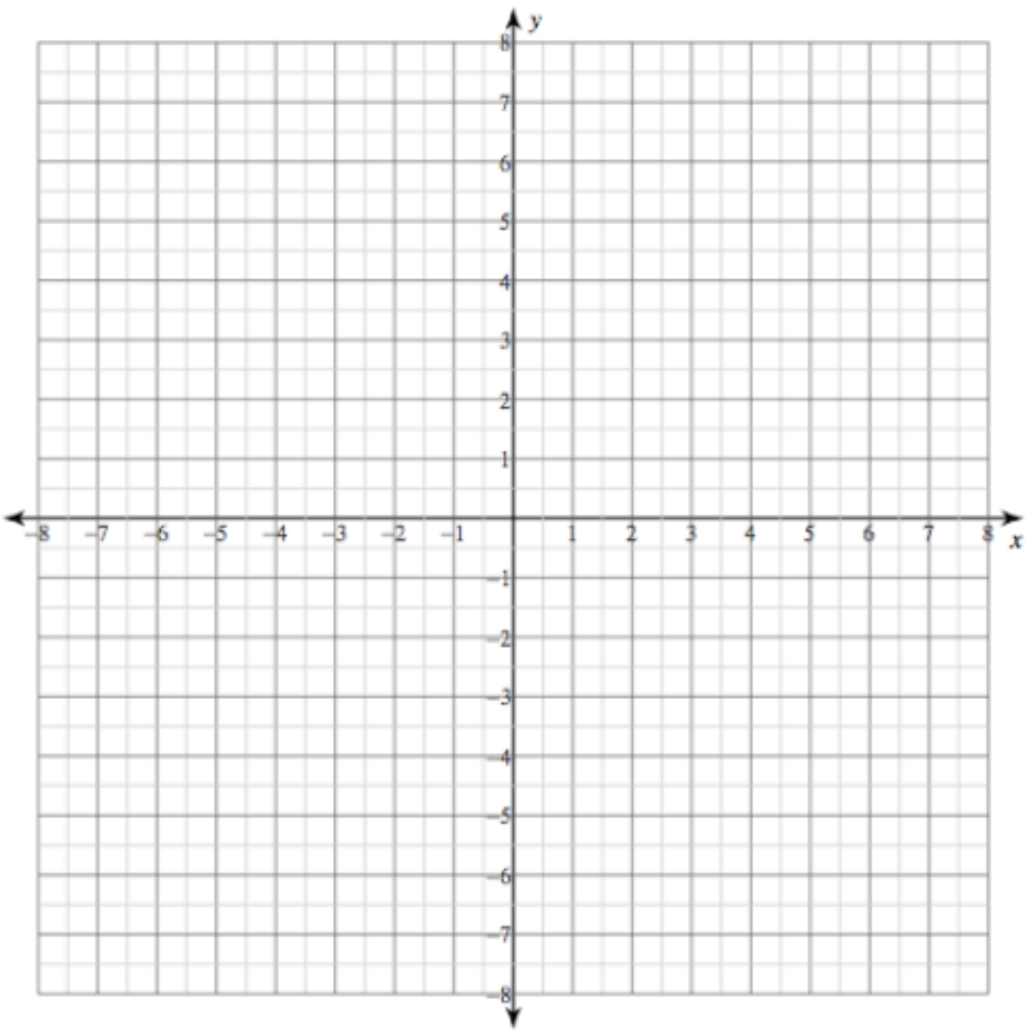




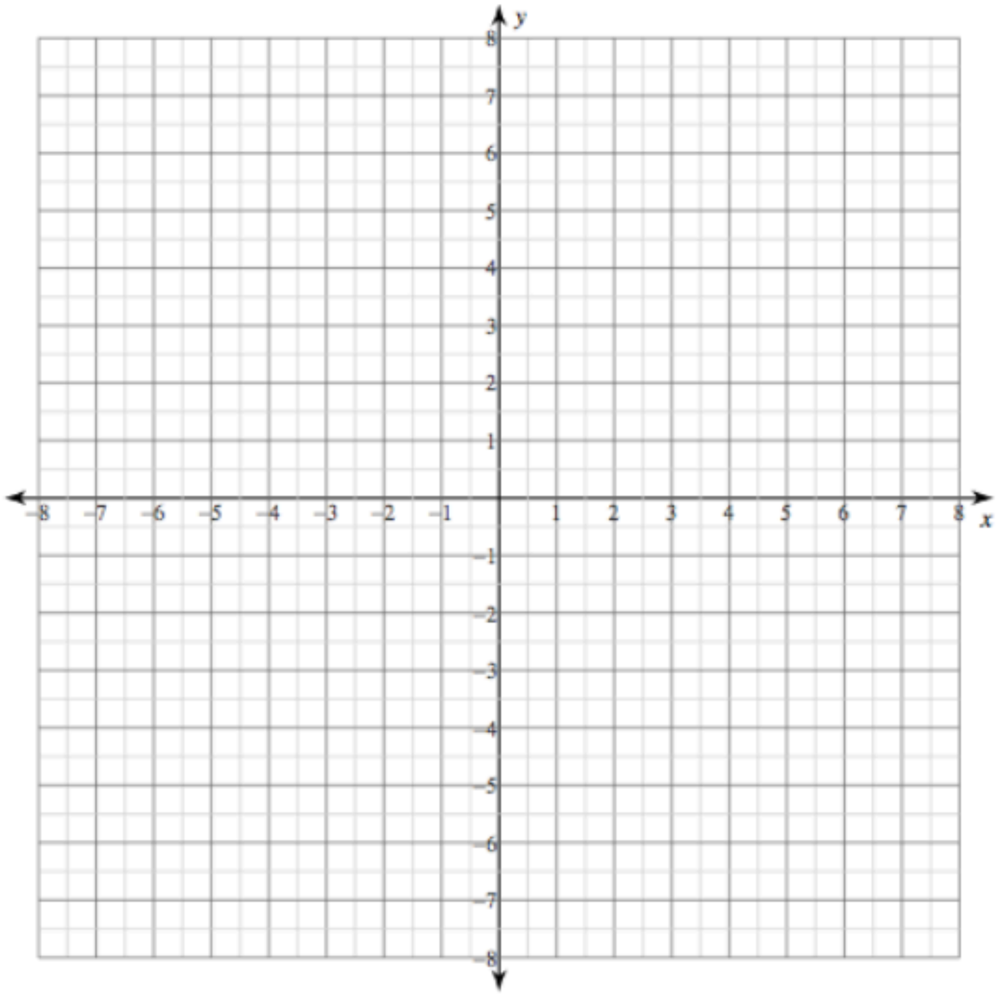
EXTENDED QUESTIONS

Find the x & y intercepts, Critical points, intervals where the function is increasing and decreasing, inflection points, intervals where the function is concave up and concave down, and relative max & min. Use this information to sketch the graph of the function.

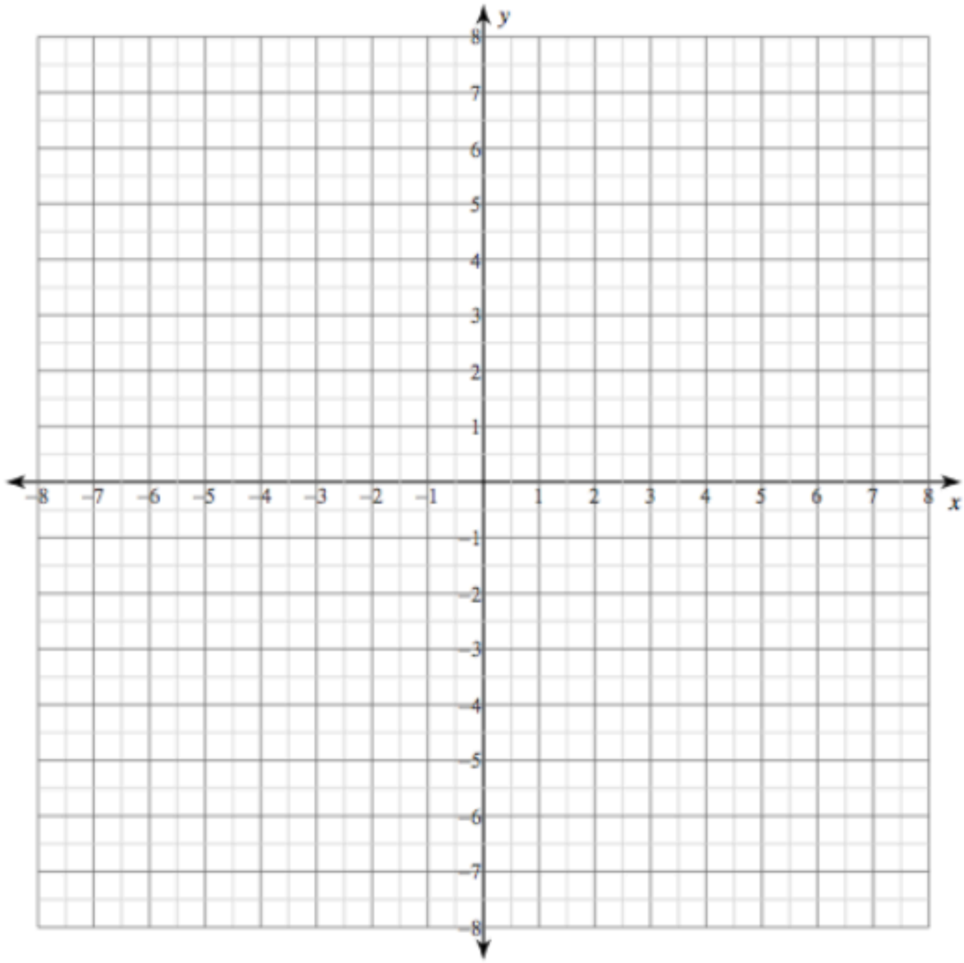
1) $y = -\frac{x^3}{3} + x^2$



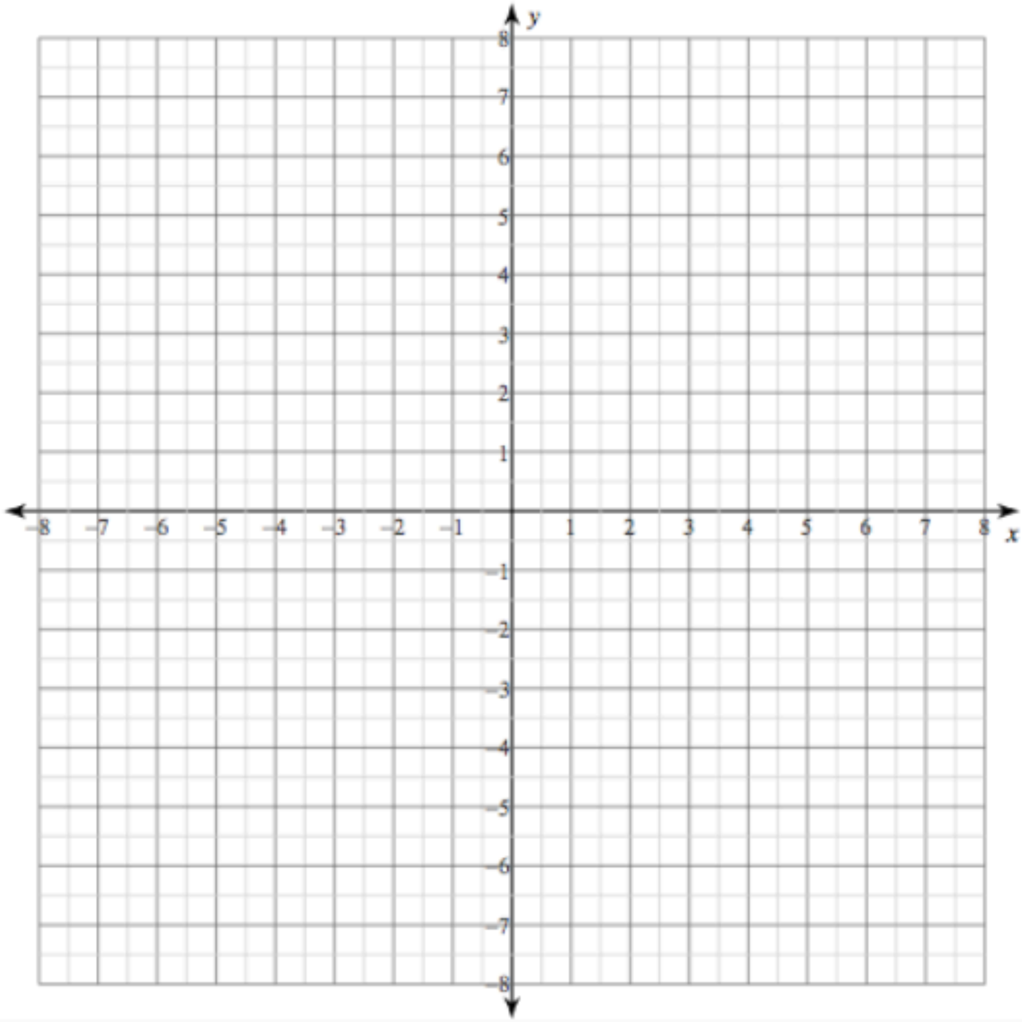
2) $$y = -\frac{x^4}{4} + x^2 - 1$$



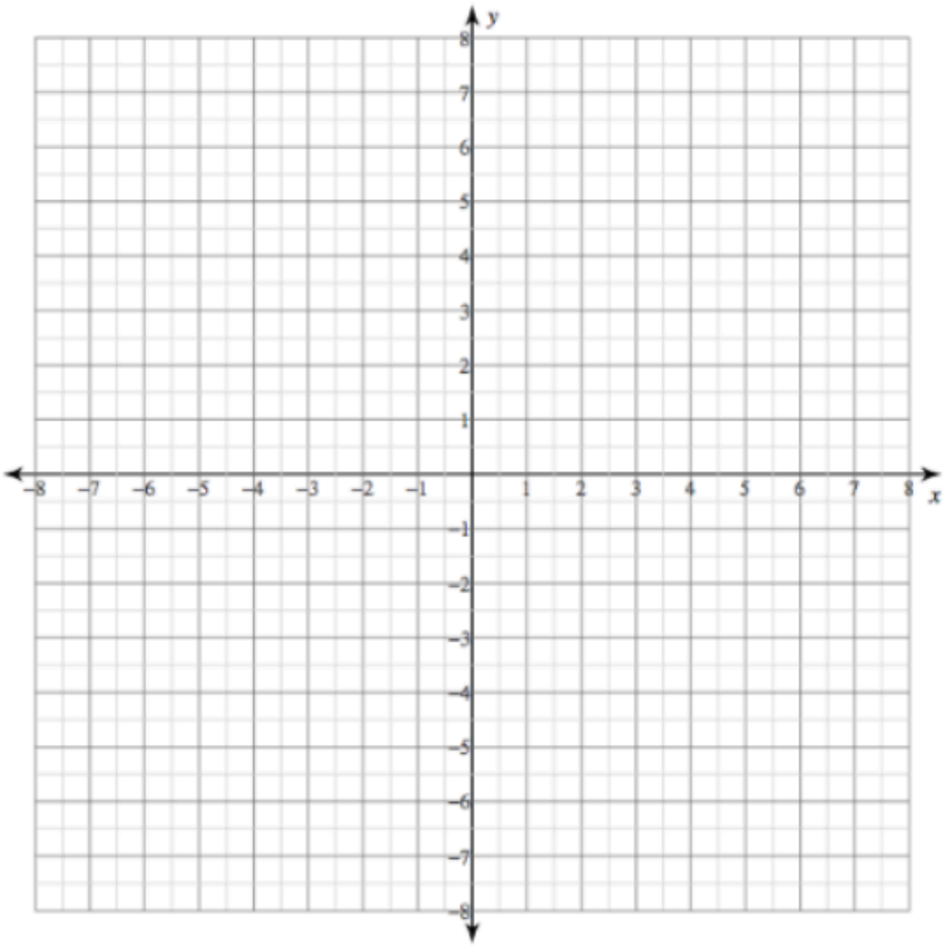
3) $y = \frac{1}{5}(x-4)^{\frac{5}{3}} + 2(x-4)^{\frac{2}{3}}$



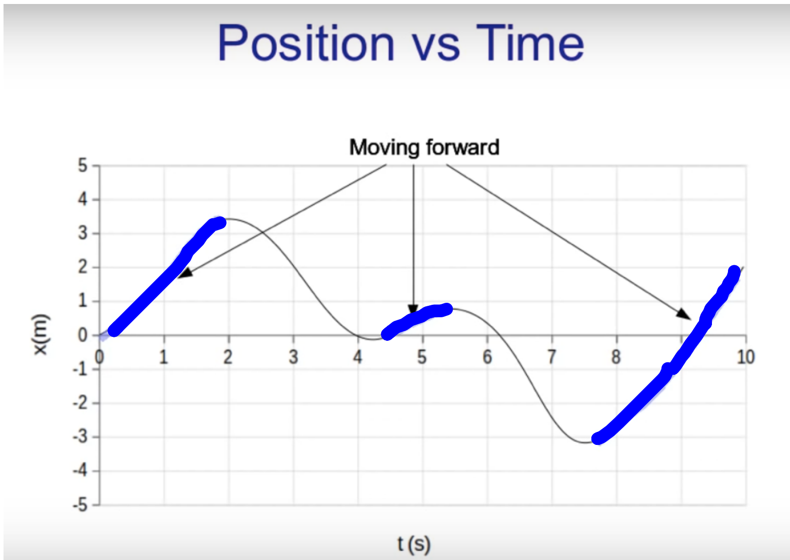
4) $y = \dfrac{7x^2 - 7}{x^3}$

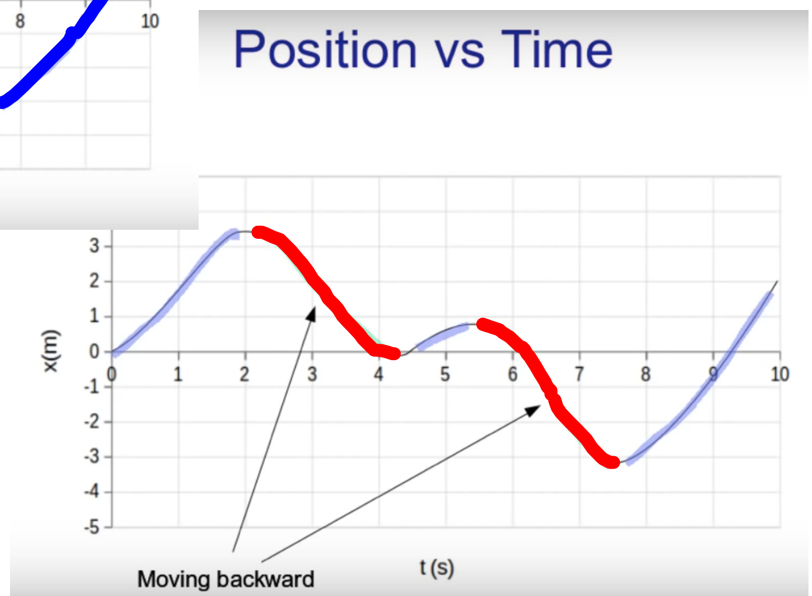


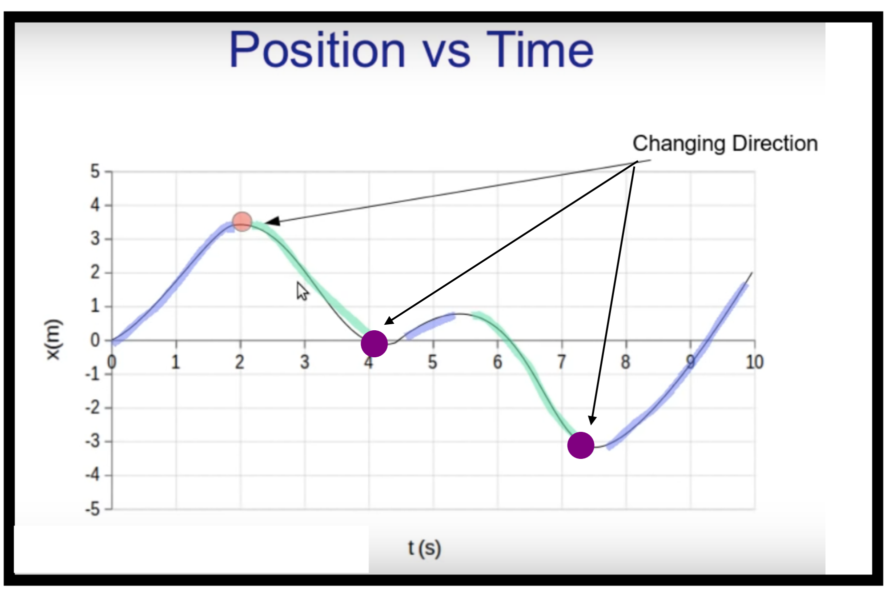
5) $y = x\ln(x)$

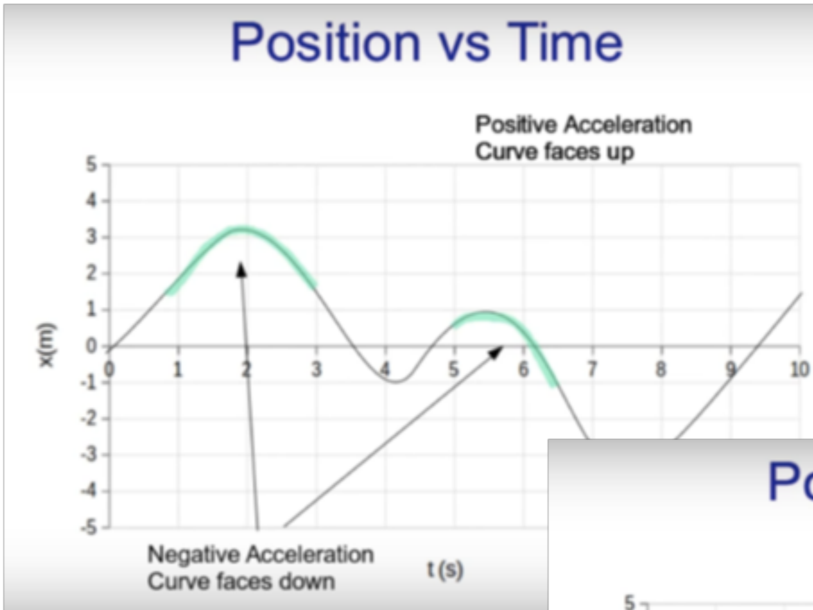


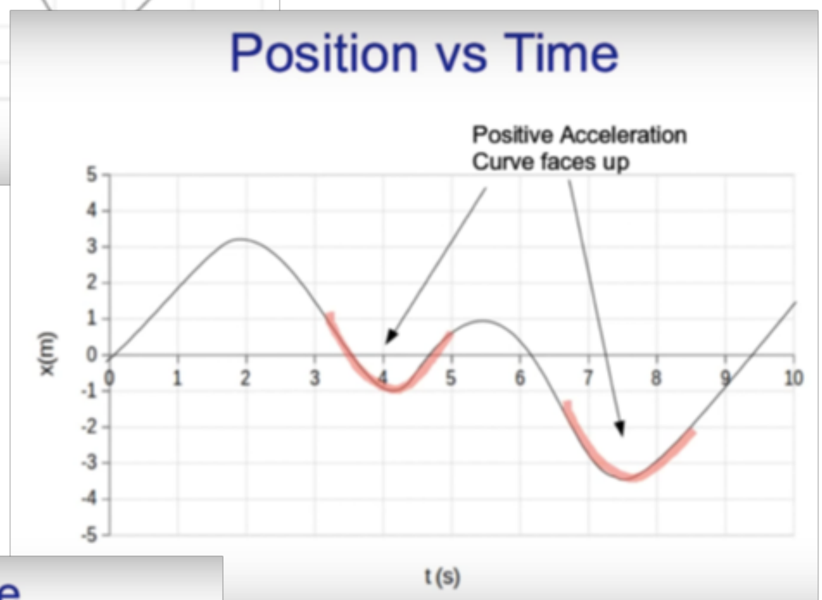
4.4 – Position, Velocity & Acceleration Graphs

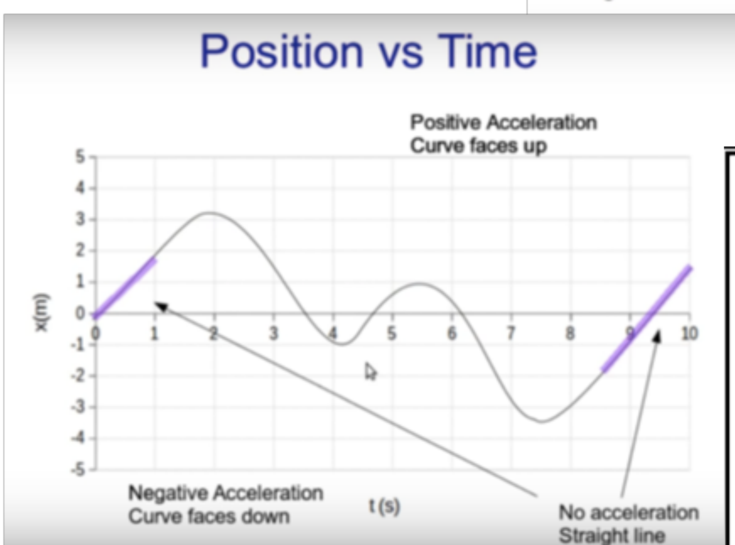


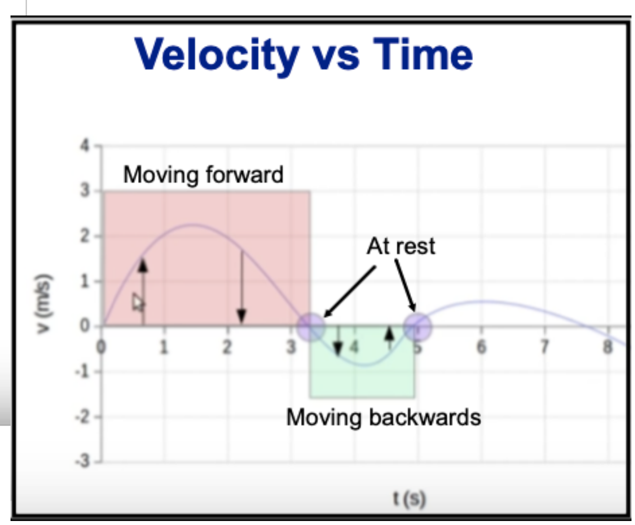




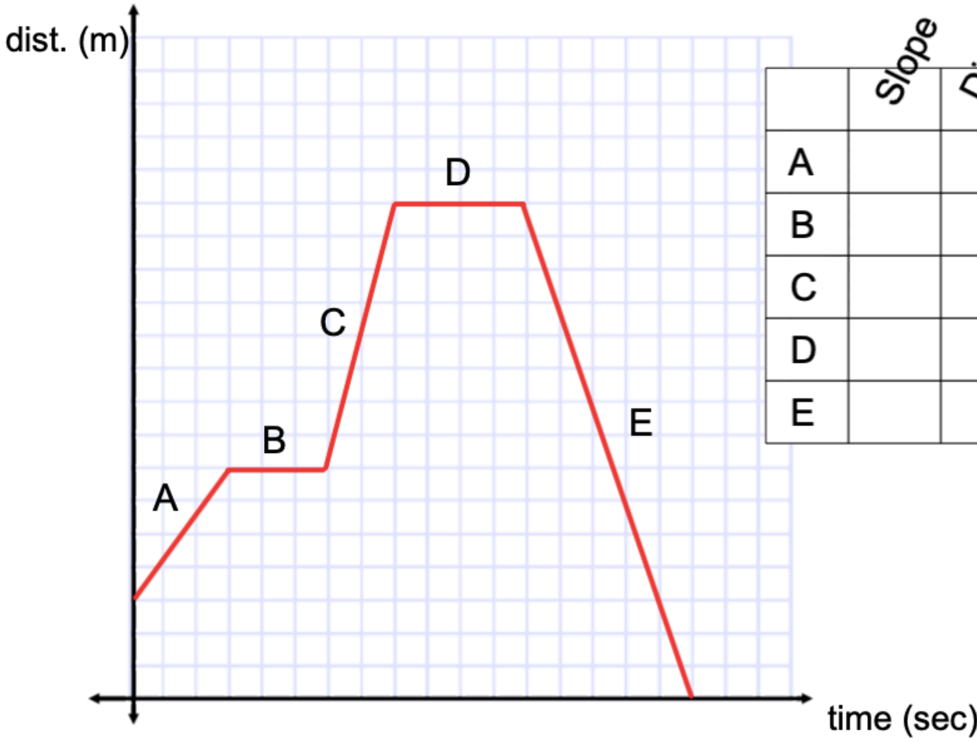

Position vs Time
Positive Acceleration
Curve faces up
x(m)
Negative Acceleration
Curve faces down
t (s)


Position vs Time
Positive Acceleration
Curve faces up
x(m)
t (s)


Position vs Time
Positive Acceleration
Curve faces up
x(m)
Negative Acceleration
Curve faces down
t (s)
No acceleration
Straight line

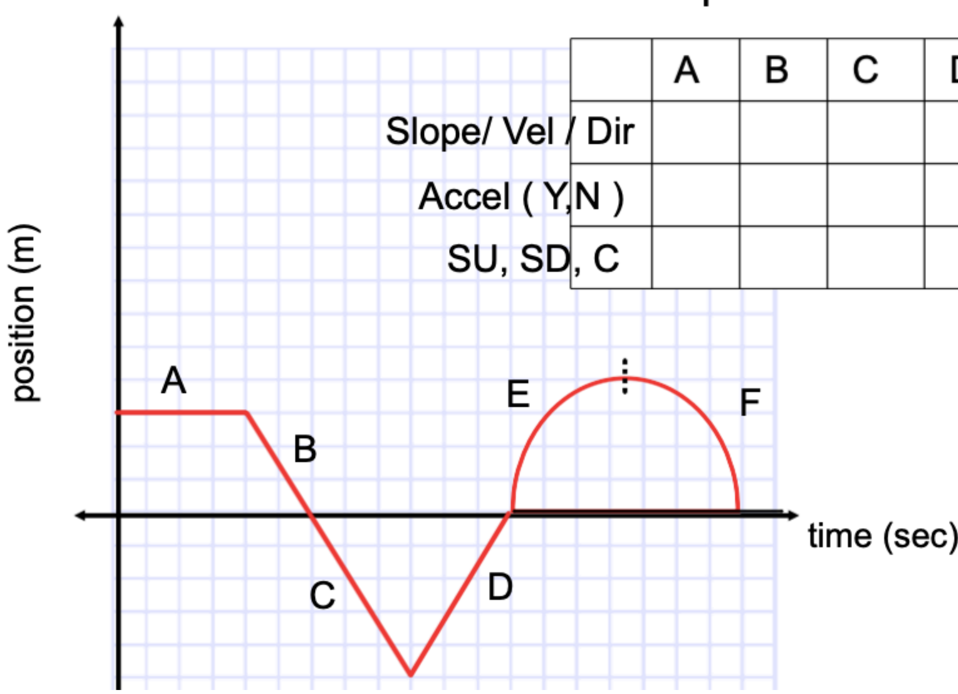

Velocity vs Time
Moving forward
At rest
v (m/s)
Moving backwards
t (s)

Position vs Time Graph



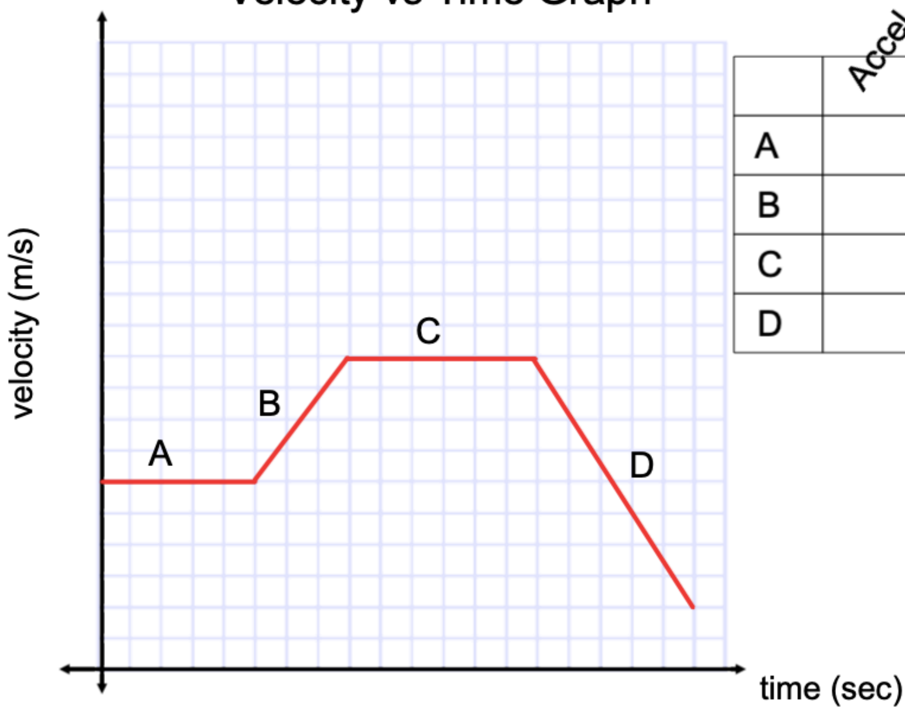
	Slope	Direction	Velocity	Speed
A				
B				
C				
D				
E				

Position vs Time Graph



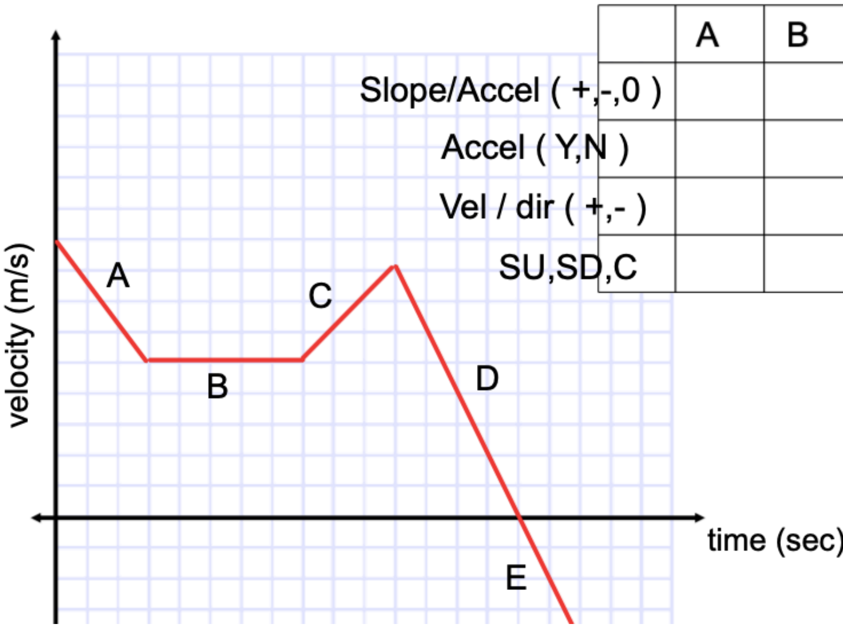
	A	B	C	D	E	F
Slope/ Vel / Dir						
Accel (Y,N)						
SU, SD, C						

Velocity vs Time Graph



	Accel	Velocity	Speed
A			
B			
C			
D			

Velocity vs Time Graph



	A	B	C	D	E
Slope/Accel (+,-,0)					
Accel (Y,N)					
Vel / dir (+,-)					
SU,SD,C					

Example:

Let $s(t) = 2t^3 + t + 2$ be the position of a particle in meters after t seconds

- What is the position at $t = 0$ and $t = 2$

- What is the velocity at $t = 0$ and $t = 2$

- What is the acceleration at $t = 0$ and $t = 2$

Example:

Let $s(t) = t^3 - 12t^2 + 45t$ be the position.

- What is the velocity when acceleration = 0 ?

- What is the total distance travelled from $t = 0$ to 4 ?

  ☆ Do Total Distance by integration

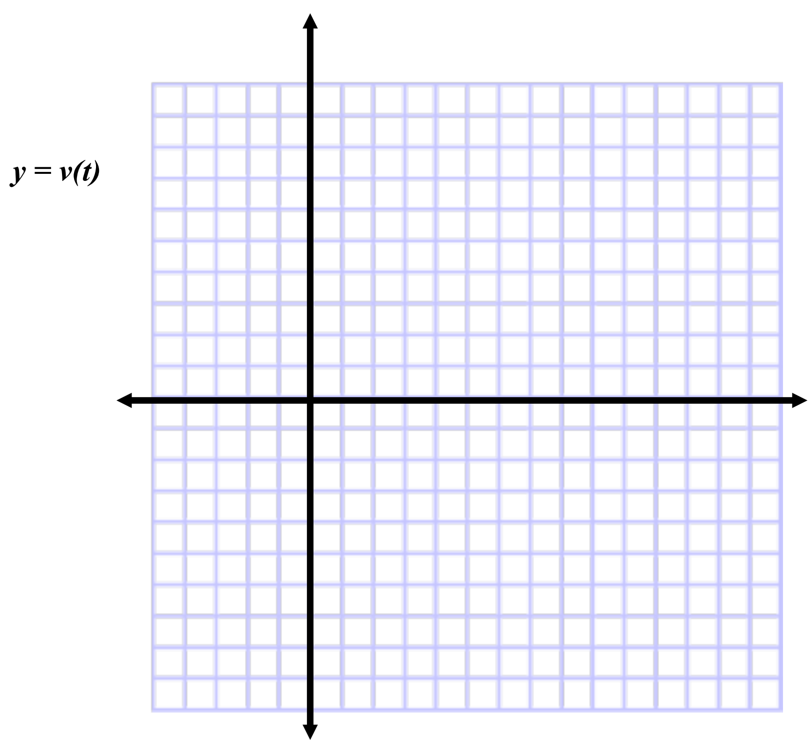
Example:

The position of a particle moving along a number line given by:

$$s(t) = t^3 - 9t^2 + 15t + 8$$

(where *t* is time in seconds & distance in metres)

1. Sketch the velocity graph. **[0, 8]**
2. Answer the following questions:
   a) What is the velocity at $t = 0$?
   b) What is the acceleration at $t = 1$?
   c) When is the particle at rest?
   d) When is the particle moving in a positive direction?
   e) When is the particle moving in a negative direction?
   f) When is the particle slowing down?
   g) When is the particle speeding up?
   h) When is the particle furthest left?
   i) When is the particle furthest right?
   j) What is the total distance travelled after 8 sec.?



You Try:

1. The position of a particle moving along a number line given by:

$$s(t) = \frac{2}{3}t^3 - 6t^2 + 10t \qquad for\ \ t \geq 0$$

where *t* is time in seconds.

Graph the position graph.

2. Then graph the velocity graph. **[0, 9]**
3. Answer the following questions.
   a) When is the particle at rest?

   b) When is the particle moving in a positive direction?

   c) When is the particle moving in a negative direction?

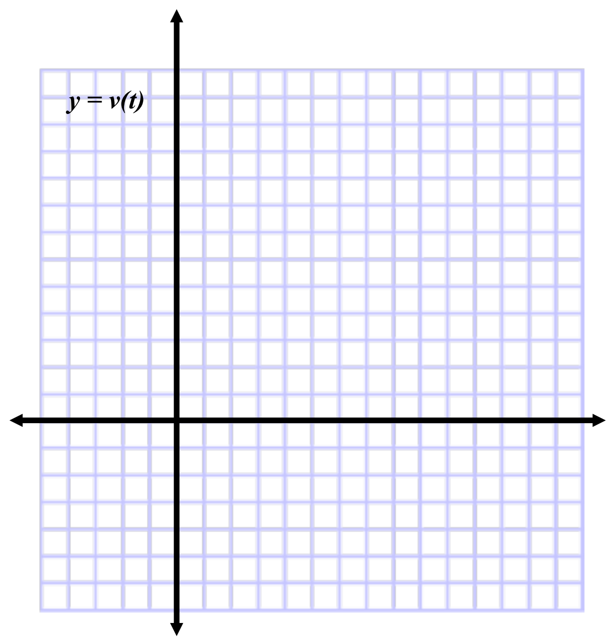
   d) When is the particle slowing down?

   e) When is the particle speeding up?

   f) What is the total distance travelled after 9 sec.?

g) When is the particle furthest left?

h) When is the particle furthest right?



4.4 - PRACTICE QUESTIONS

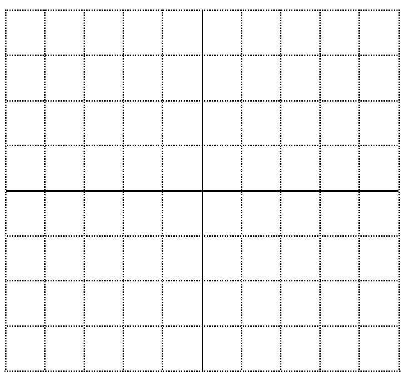
1. For the following situations, state whether the velocity is positive or negative and whether the speed is increasing or decreasing.

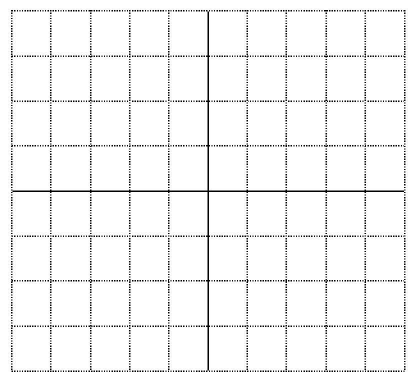
	Velocity + or –	*Speed I or D*
a) braking while moving forward		
b) accelerating while in reverse gear		
c) accelerating while in forward gear		
d) braking while moving backward		

2. *A particle moves in a straight line with a position function s(t), t $\geq$ 0. In each case sketch the graphs of the velocity:*

$a)\ \ s(t) = 2t^3 - 3t^2 + 6t$

$b)\ \ s(t) = t^2 - 4t + 4$





3. *For each function above complete the following chart:*

	$a)\ \ s(t) = 2t^3 - 3t^2 + 6t$	$b)\ \ s(t) = t^2 - 4t + 4$
a) when is the particle at rest		
b) when is the particle moving in a positive direction		
c) when is the particle moving in a negative direction		
d) when is the speed of the particle increasing		
e) when is the speed of the particle decreasing		
f) what is the total distance traveled after 6 sec of motion, starting t = 0		

4.

Given the position equation:

$$s(t) = \frac{1}{3}t^3 - 4t^2 + 12t + 2\,; \quad [0, 7]$$

a) Sketch a graph of velocity

b) Answer the following questions:

I. When is the particle at rest?

II. When is the particle moving in a positive direction?

III. When is the particle moving in a negative direction?

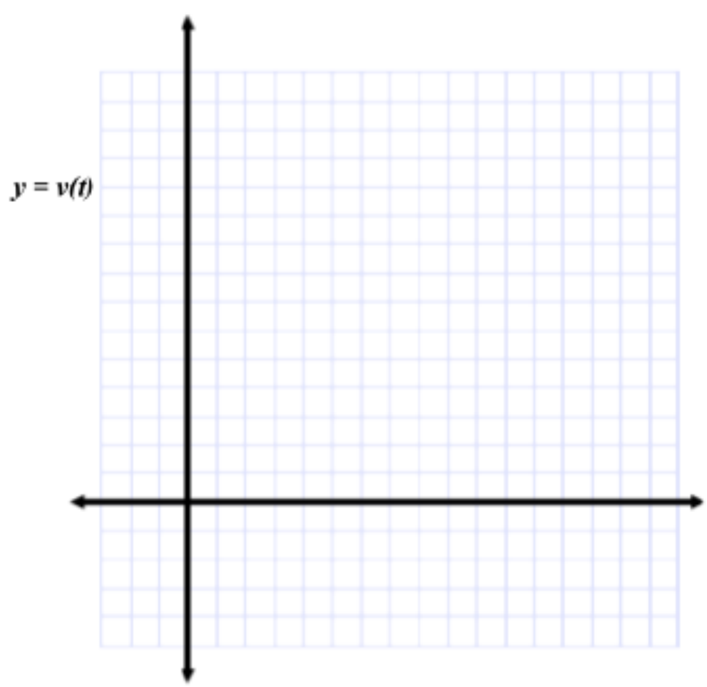
IV. When is the particle slowing down?

V. When is the particle speeding up?

VI. What is the total distance travelled after 7 sec.?

VII. When was the particle furthest left?

VIII. When was the particle furthest right?



5.

Given the position equation:

$$s(t) = \frac{2}{3}t^3 - 6t^2 + 10t\ ;\ \ [0, 7]$$

a) Sketch a graph of velocity

b) Answer the following questions:

I. When is the particle at rest?

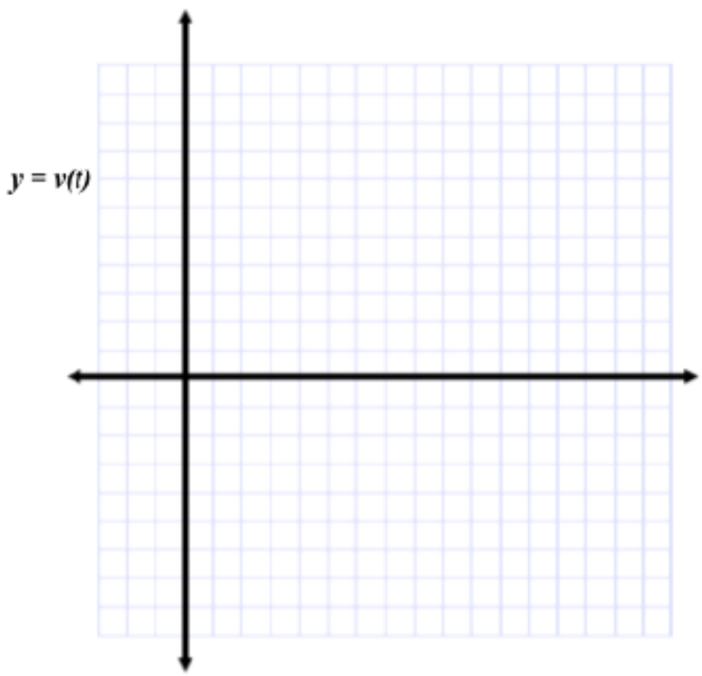
II. When is the particle moving in a positive direction?

III. When is the particle moving in a negative direction?

IV. When is the particle slowing down?

V. When is the particle speeding up?

VI. What is the total distance travelled after 7 sec.?



6.

Given the position equation:

$s(t) = t^3 - 4t^2 + 3t$; $[0, 4]$

a) Sketch a graph of velocity

b) Answer the following questions:

I. When is the particle at rest?

II. When is the particle moving in a positive direction?

III. When is the particle moving in a negative direction?

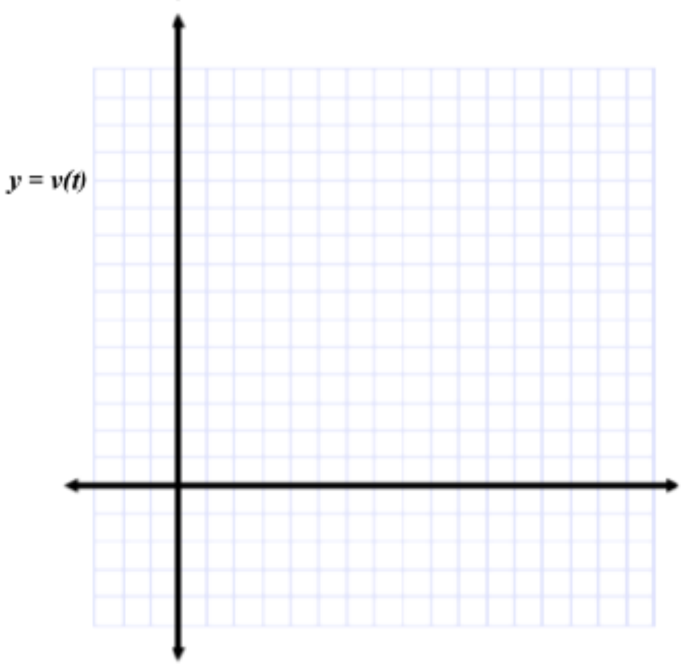
IV. When is the particle slowing down?

V. When is the particle speeding up?

VI. What is the total distance travelled after 3 sec.?

VII. When was the particle furthest left?

VIII. When was the particle furthest right?

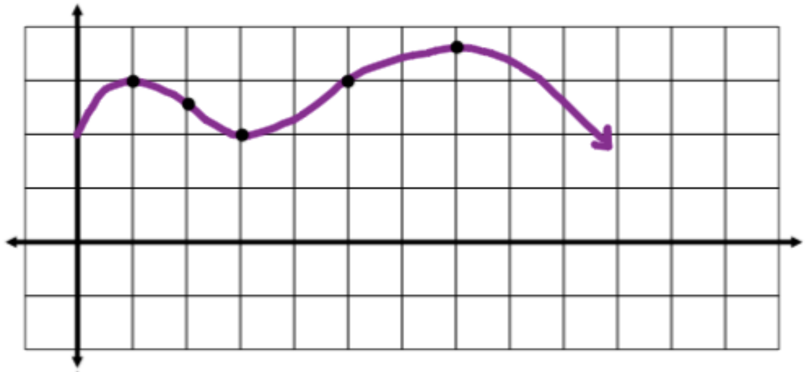


***BE SURE TO DO MULTIPLE CHOICE QUESTIONS ONLINE **VERSION 7** #77 - 91

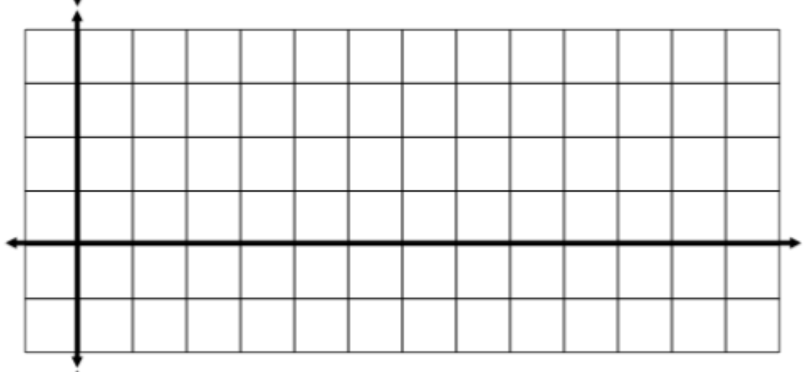
4.5 - SKETCHING DERIVATIVES

STEPS:

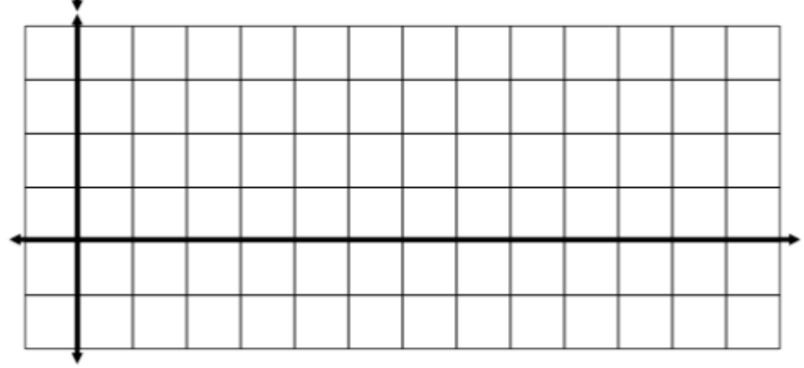
s(t)



v(t)



a(t)

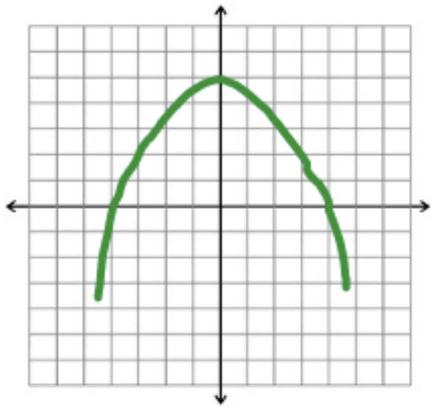


Example:

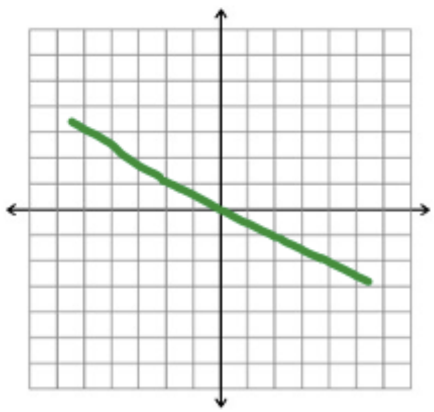
Given the graphs of the first and second derivative of a function *f(x)*, sketch the graph of *f(x)* passing through the indicated points.

STEPS:

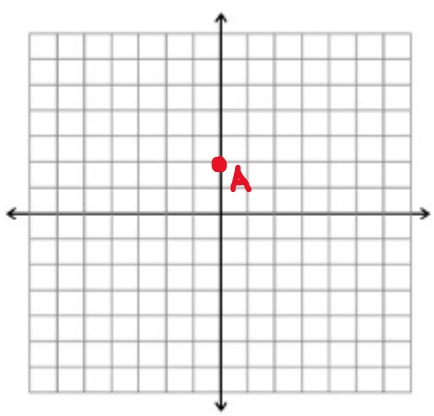
y'



y''



y

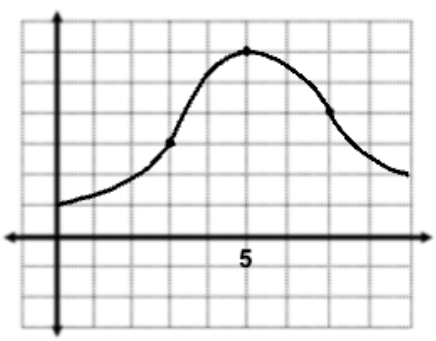


4.5 - PRACTICE QUESTIONS

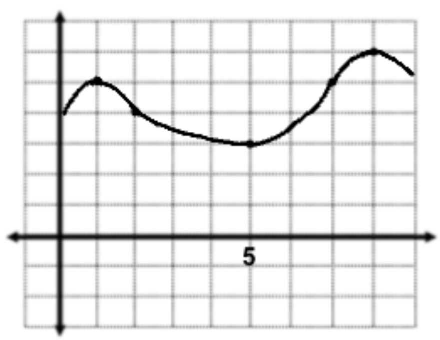
1. *Answer the following questions using the graphs below:*

	Graph A	*Graph B*
a) at what times is the velocity 0		
b) at what times is the object moving in the a positive or negative direction		
c) at what times is the acceleration 0		
d) at what times is the acceleration + or -		
e) when is the object slowing down or speeding up		

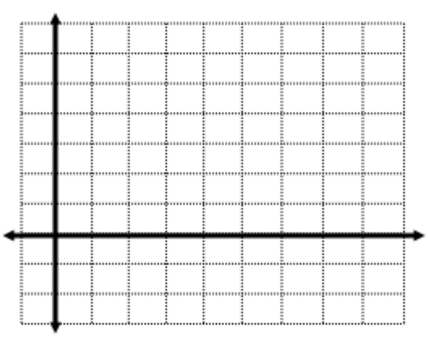
GRAPH A

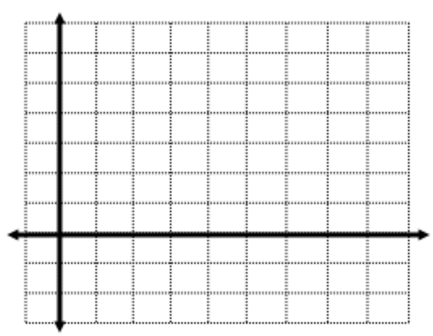


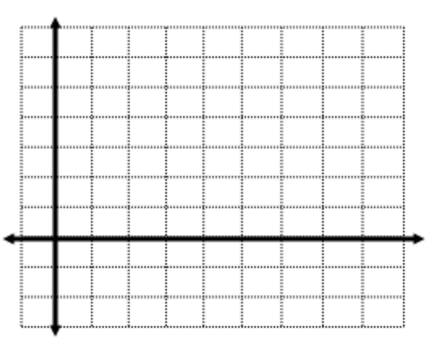
GRAPH B

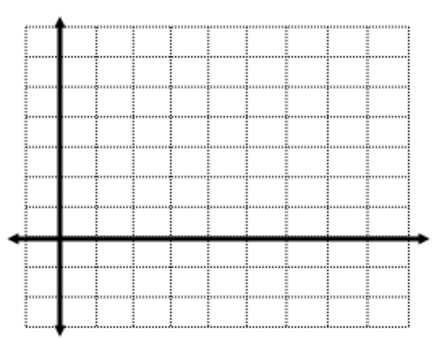


2. *Make a rough sketch of the velocity and acceleration in the above:*



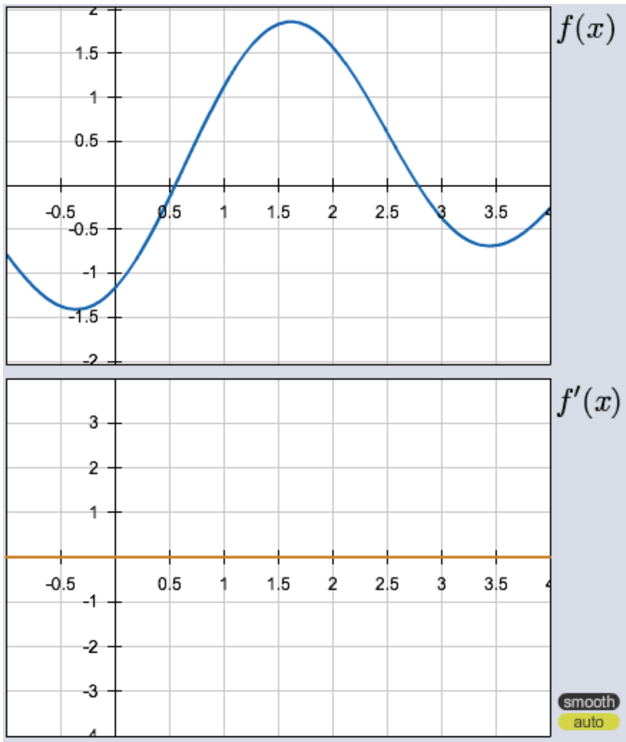




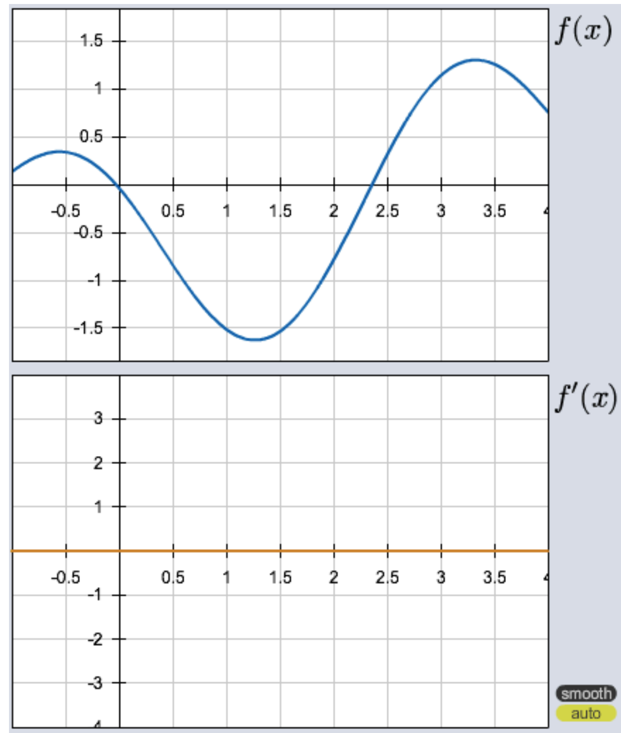


3. Given *f(x)* Sketch *f'(x)*.

a)

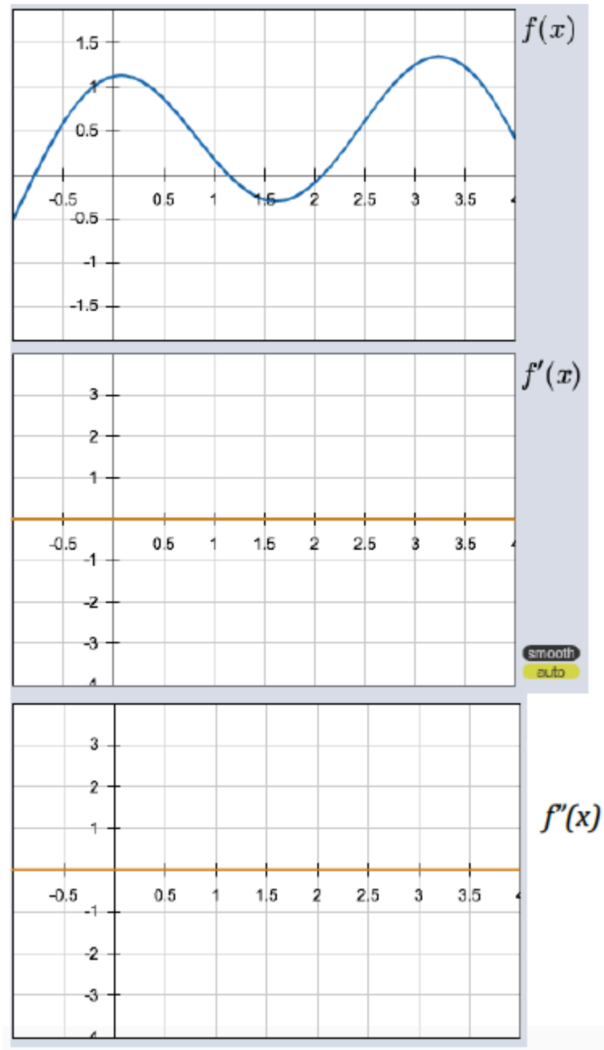


b)

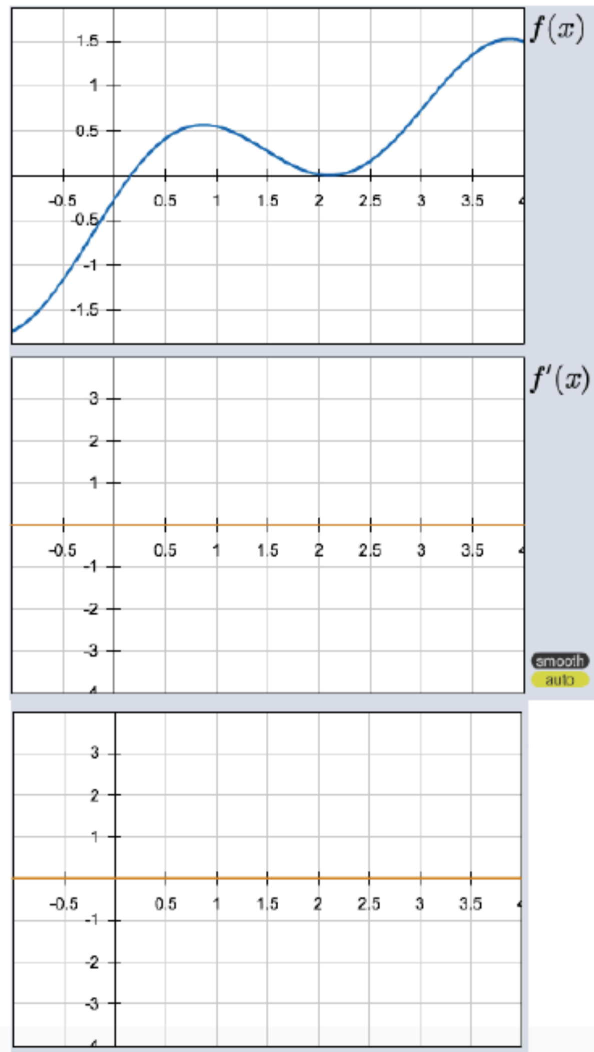


4. Given *f(x)* Sketch *f'(x)* and *f''(x)*

a)

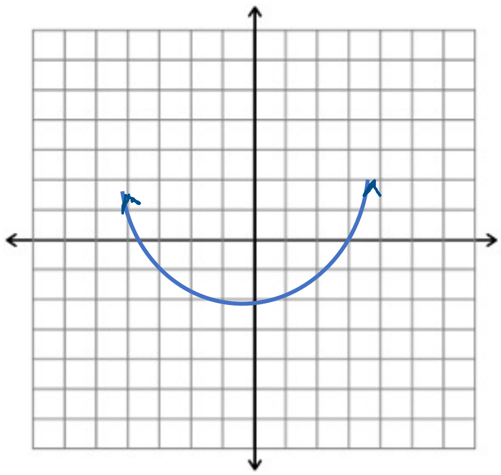


b)

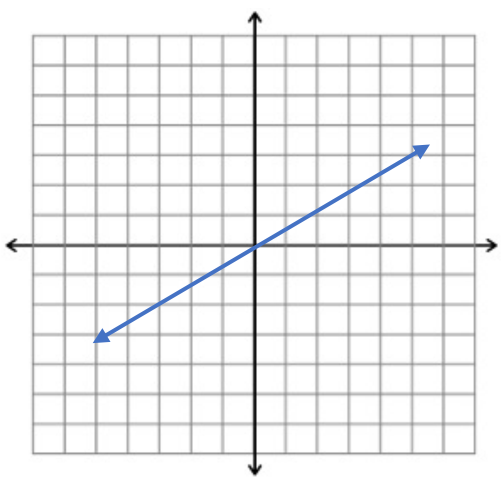


5. Sketching Position from Velocity & Acceleration Graphs that goes through Point A and/or B.

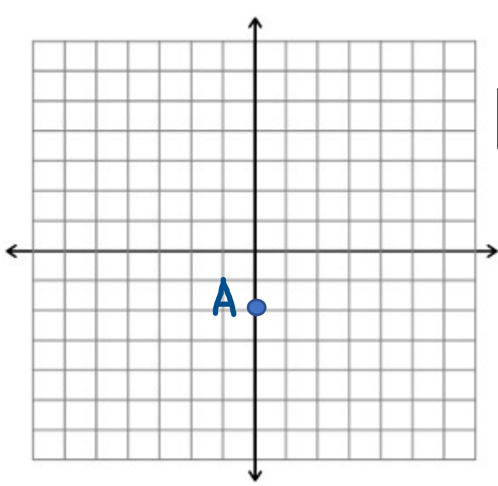
a)



y'

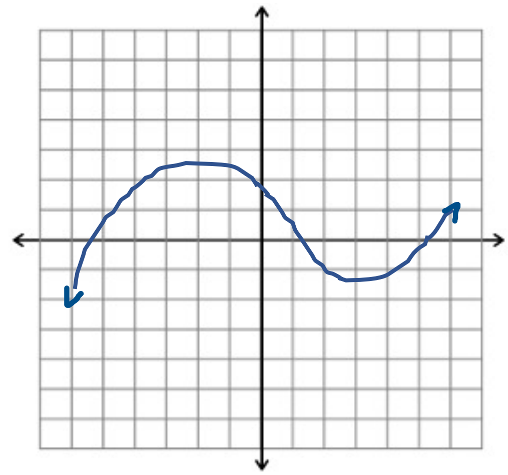


y''

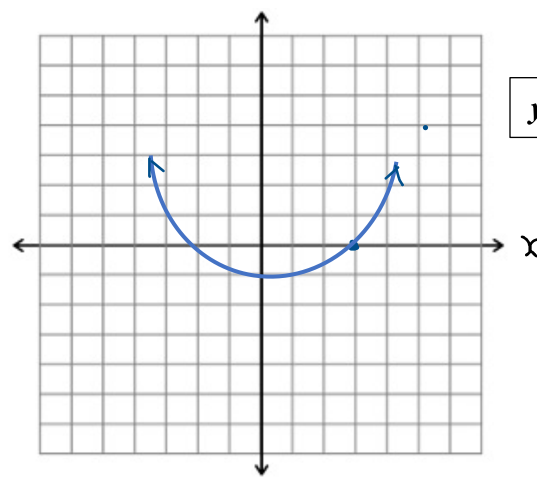


y

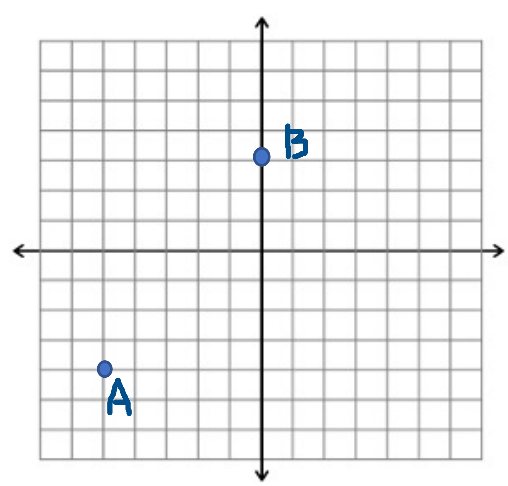
b)



y'

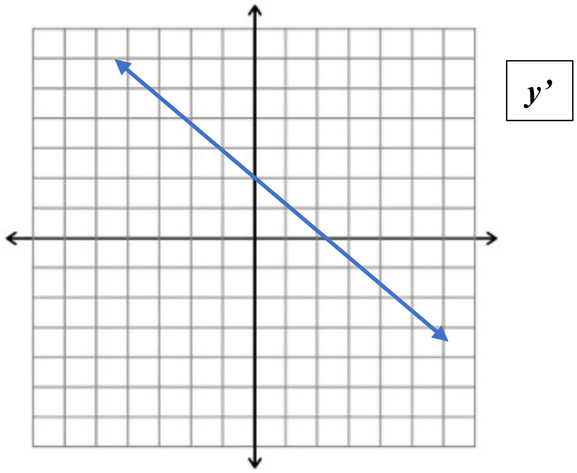


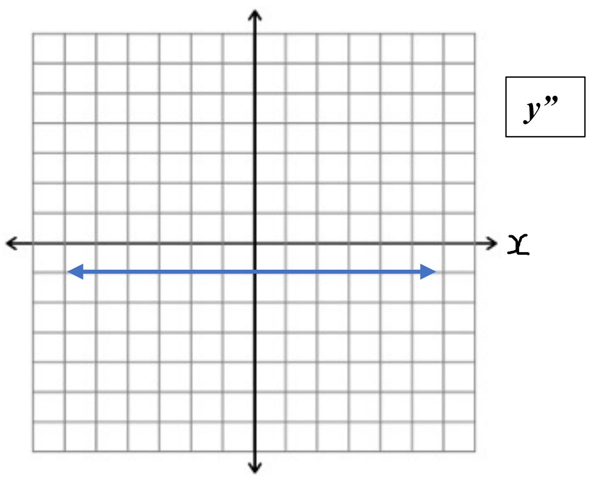
y''

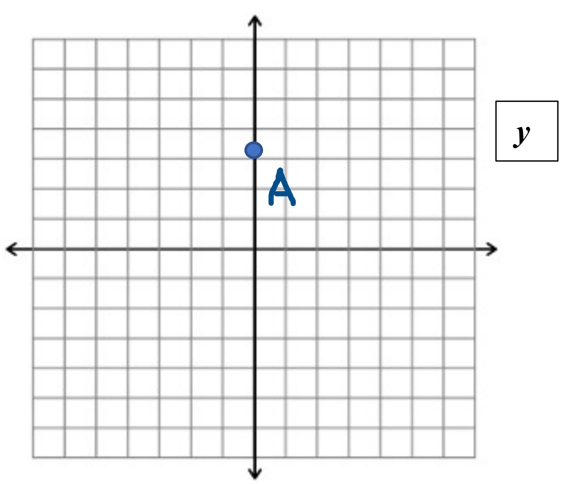


y

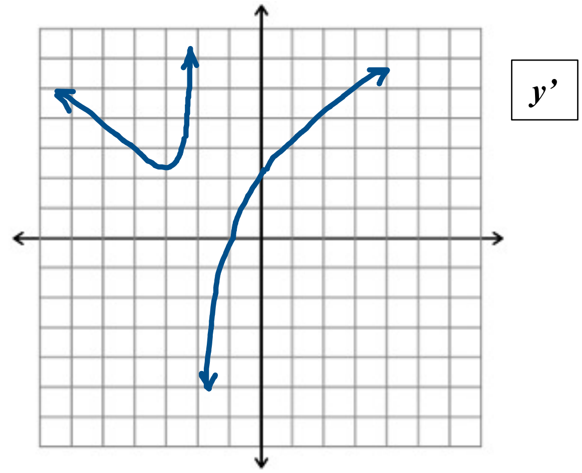
c)

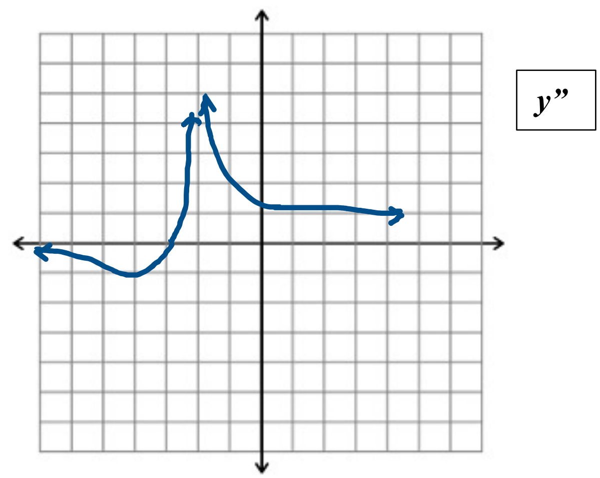


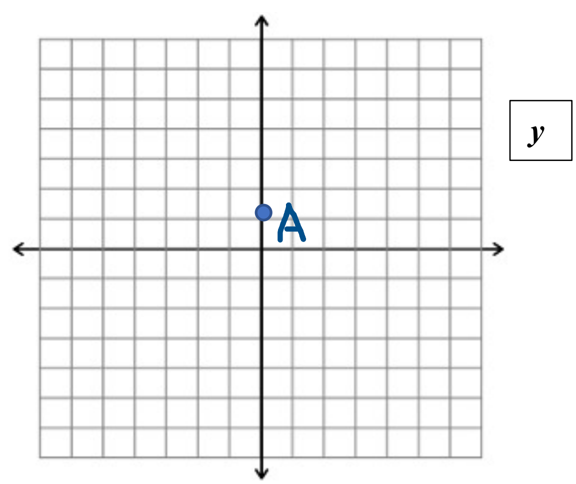




d)







4.6 - Sketch a Function with Conditions:

Example

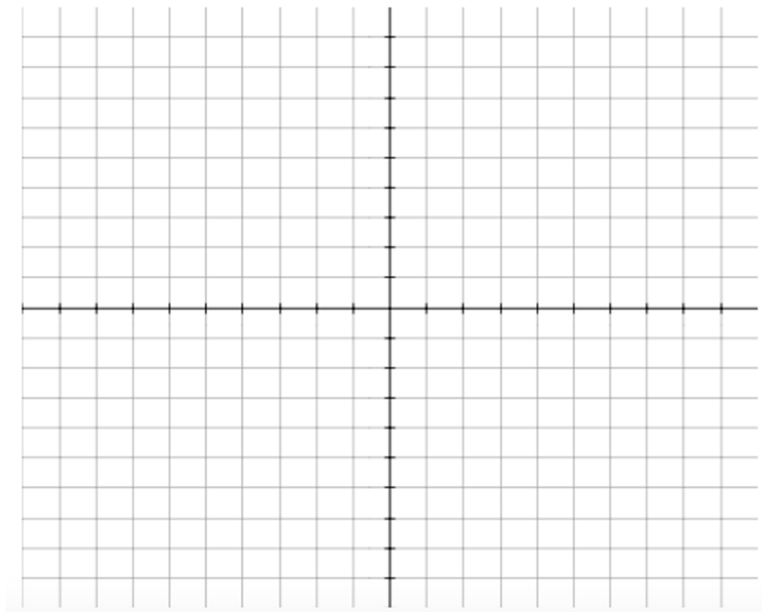
Sketch a graph of a function satisfying each of the following conditions:

$f(-2)$ *is undefined*
$f(3)$ *is undefined*
$x-int\ at-1\ \&-4$

$\lim_{x\to\infty} f(x) = -1$
$\lim_{x\to-\infty} f(x) = -1$
$\lim_{x\to 2^+} f(x) = -1$
$\lim_{x\to 2^-} f(x) = 1$

$f'(-2)$ *is undefined*
$f'(3)$ *is undefined*
$f'(1) = 0$
$f'(-5) = 0$

$f''(-2)$ *is undefined*
$f''(3)$ *is undefined*
$f''(-6) = 0$



Interval		$(-\infty,-5)$	$(-5,-2)$	$(-2,1)$	$(1,3)$	$(3,\infty)$
Sign of $f'(x)$		$+$	$-$	$+$	$-$	$+$

Interval		$(-\infty,-6)$	$(-6,\infty)$
Sign of $f''(x)$		$+$	$-$

4.6 - PRACTICE QUESTIONS

1. *Sketch a graph of a function satisfying each of the following conditions:*

a)

$f(0)$ is undefined
$f(1) = 0$

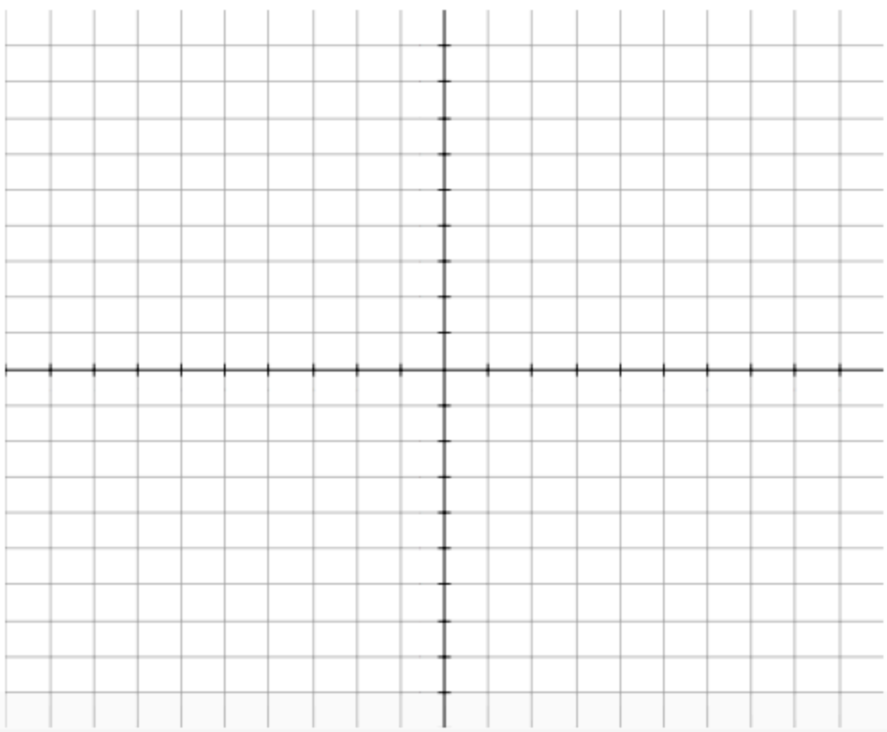
$\lim\limits_{x \to \infty} f(x) = 0$
$\lim\limits_{x \to -\infty} f(x) = 0$

$f'(2) = 0$

$f''(3) = 0$

Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
Sign of $f'(x)$	-	+	-

Interval	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
Sign of $f''(x)$	-	-	+



b)

$f(1)$ is undefined

$\lim\limits_{x\to\infty} f(x) = 2$

$\lim\limits_{x\to-\infty} f(x) = 2$

$f'(1)$ is undefined

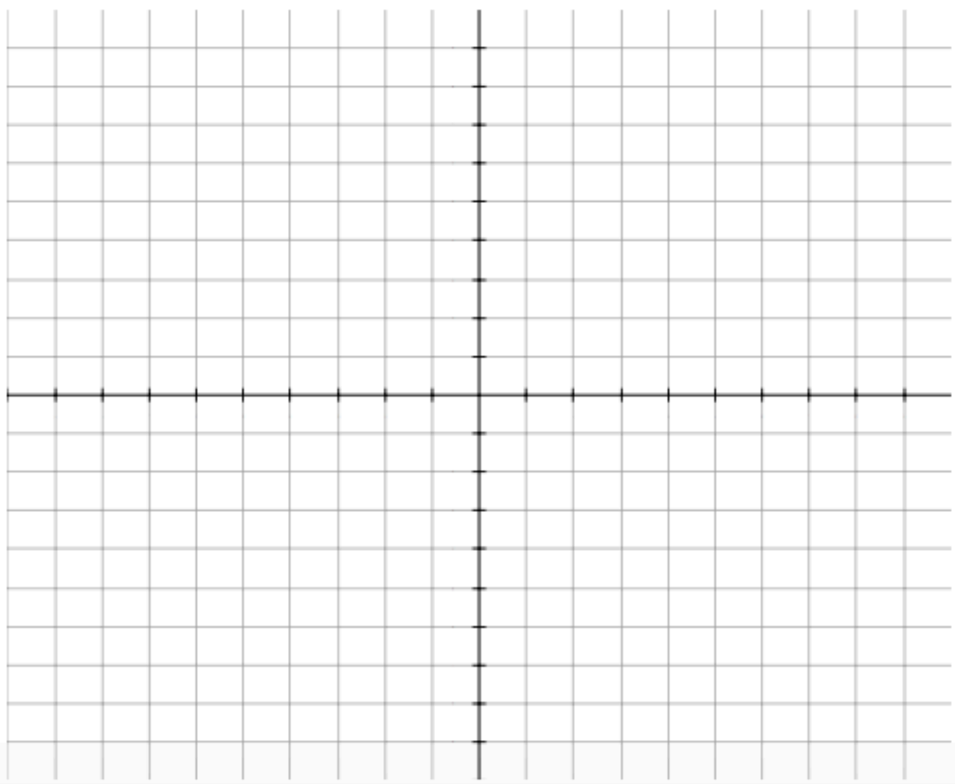
$f''(1)$ is undefined

$f''(2) = 0$

$f''(4) = 0$

Interval	$(-\infty,1)$	$(1,2]$	$[2,\infty)$
Sign of $f'(x)$	-	-	-

Interval	$(-\infty,1)$	$(1,2]$	$[2,4]$	$[4,\infty)$
Sign of $f''(x)$	-	+	-	+



c)

$f(0) = -3$
$f(-3) = 0$
$f(4) = 0$

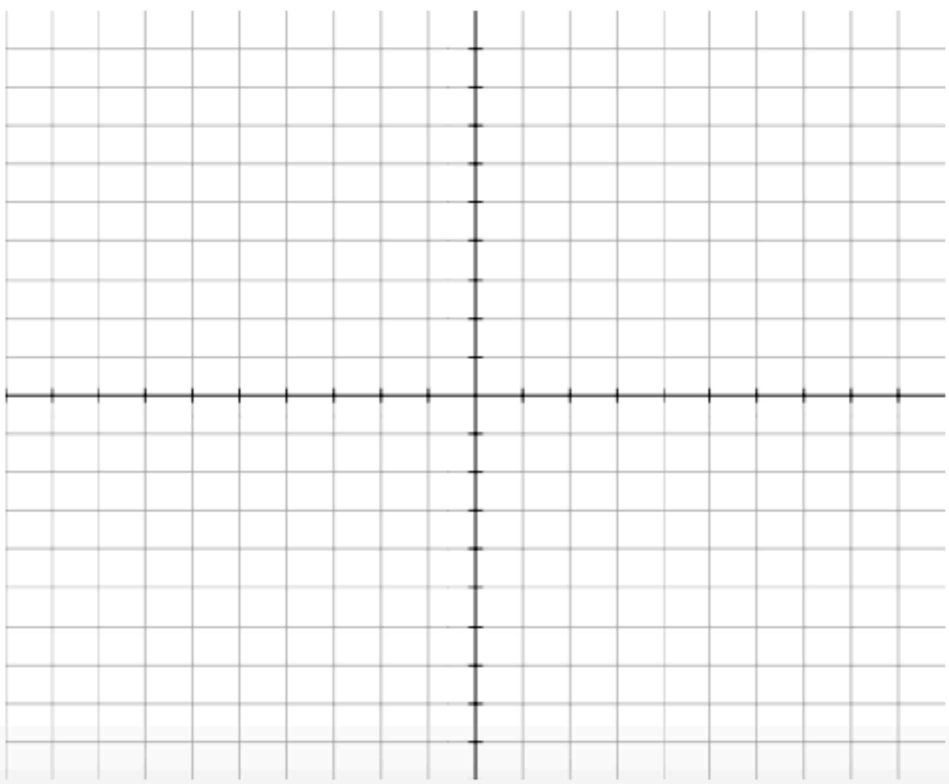
$\lim_{x \to \infty} f(x) = -\infty$
$\lim_{x \to -\infty} f(x) = -\infty$

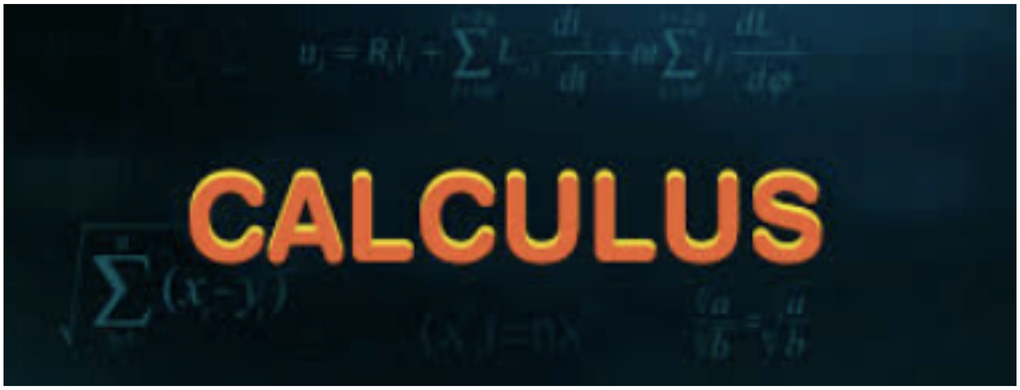
$f'(-3) = 0$
$f'(1) = 0$
$f'(4) = 0$

$f''(-1) = 0$
$f''(3) = 0$

Interval	$(-\infty, -3]$	$[-3, 1]$	$[1, 4]$	$[4, \infty)$
Sign of $f'(x)$	+	-	+	-

Interval	$(-\infty, -1]$	$[-1, 3]$	$[3, \infty)$
Sign of $f''(x)$	-	+	-





SEMINAR NOTES

UNIT 5

Review of Logarithms

Rules

$\log AB = \log A + \log B$

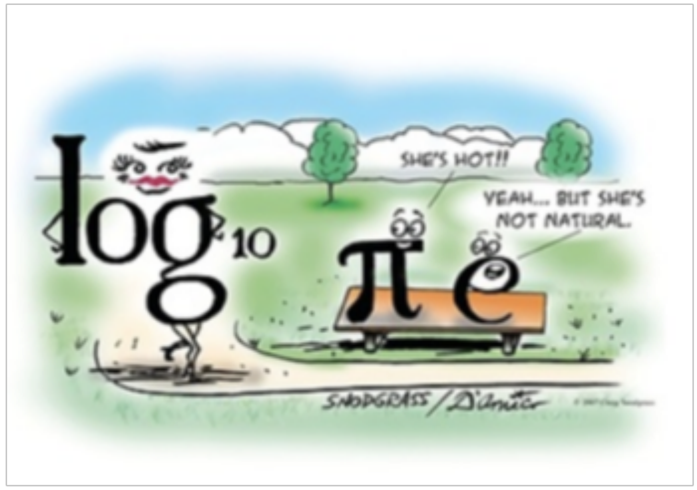
$\log \frac{A}{B} = \log A - \log B$

$\log A^B = B \log A$

$\log_A A = 1$

$\log_A 1 = 0$

$\ln = \log_e$



Example 1

Solve exactly:

a) $\ln(x) + 1 = 0$

b) $3\ln(x) - 8 = 0$

c) $(\ln(x))^2 + (\ln(x)) = 12$

d) $e^{2x} + 2e^x = 15$

e) $e^x - 8e^{-x} = 2$

Example 2

Simplify: $\ln(4e^x)$

Tangent and Normal Equation of Lines

The derivative of a function has many applications to problems in calculus. It may be used in curve sketching; solving maximum and minimum problems; solving distance; velocity, and acceleration problems; solving related rate problems; and approximating function values.

The derivative of a function at a point is the slope of the tangent line at this point. The **normal line** is defined as the line that is perpendicular to the tangent line at the point of tangency. Because the slopes of perpendicular lines (neither of which is vertical) are negative reciprocals of one another, the slope of the normal line to the graph of $f(x)$ is $-1/f'(x)$.

Example:

Find the equation of Tangent and Normal line to the curve and point given.

$a)\quad y = x^4 + 2\ln(x) \quad at \quad (e, 2)$

$b)\quad y = 2\cos(x) \quad at \quad \left(\frac{3\pi}{2}, 0\right)$

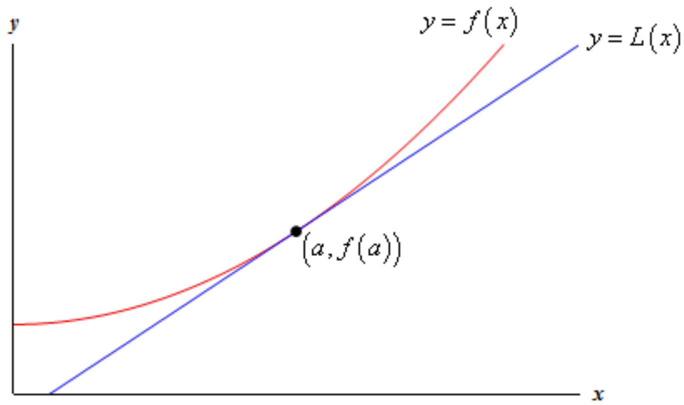
5.1 - Tangent or Linear Approximation

Here we're going to take a look at an application not of derivatives but of the tangent line to a function. Of course, to get the tangent line we do need to take derivatives, so in some way this is an application of derivatives as well.

Given a function, $f(x)$, we can find its tangent at $x = a$. The equation of the tangent line, which we'll call $L(x)$ for this discussion, is,

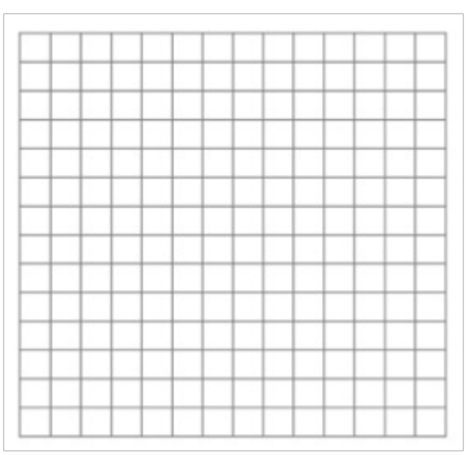
$$L(x) = f(a) + f'(a)(x - a)$$

Take a look at the following graph of a function and its tangent line.



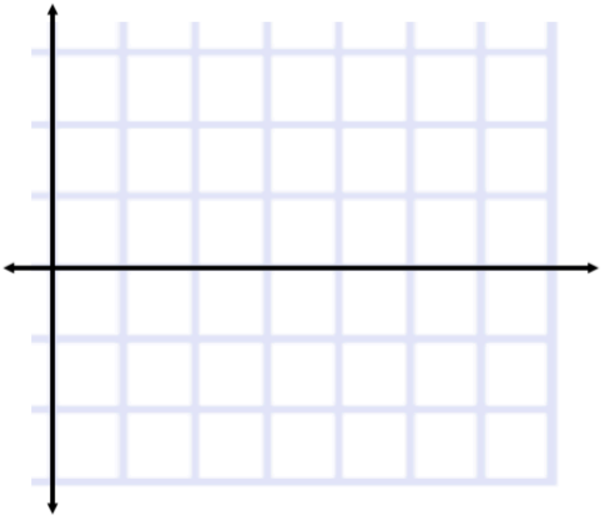
Example:

Find the Tangent Approximation to the function $f(x) = \sqrt{x}$ at $x = 4$ to find the square root of 3.



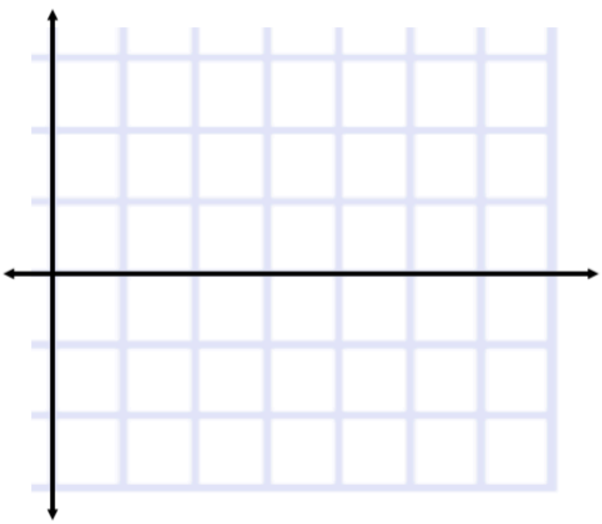
Newton's Method to Solve Equations

A method used to find the x-int through a process of ITERATIONS.



$$x_1$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$\vdots$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example: Solve the equation $x^2 - 8 = 0$ using x-initial value $x = 3$.



You Try: Solve the equation $2x^3 - 8x^2 + 5x + 2$ using x-initial value $x = 1$.

5.1 - PRACTICE QUESTIONS

1. Solve each of the following equations exactly:

a) $\ln x + 3 = 0$

b) $2 \ln x - 9 = 0$

c) $(\ln x)^2 - 4 = 0$

d) $(\ln x)^2 + (\ln x) - 2 = 0$

e) $2 \ln x = \ln 16$

f) $2 \ln x = \ln(4x + 5)$

2. Simplify:

a) $\ln(e^{-2})$

b) $\ln(e^{-2}) + \ln(e^{3})$

c) $e^{(2 \ln 3)}$

d) $e^{(-2 \ln 3)}$

e) e^-$\ln(\frac{1}{x})$

f) $\ln(8e^{3})$

3. Solve each of the following equations <u>exactly:</u>

a) $e^{2x} + e^{x} - 6 = 0$

b) $e^{2x} - 2e^{x} = 8$

c) $e^{x} + 4e^{-x} = 5$

d) $e^{x} = 6e^{-x} + 1$

4. Differentiate each function:

a) $y = (\ln x)^2 + \ln (x^2)$

b) $y = (x \ln x)^2$

c) $y = 2e^{\left(x^2 - x\right)}$

d) $y = 2xe^{(2x)}$

e) $y = \ln (\pi + e^{(2x)})$

f) $y = \dfrac{e^x}{\ln x}$

g) $y = \ln (\sin y) + x^2$

h) $y = e^{(2y)} + xy$

5. *Find the equation of the tangent line and of the normal line to the curve $y = \ln x$ at the point $(e, 1)$.*

6. *Find the equation of the tangent line and of the normal line to the curve $y = e^x$ at the point $(2, e^2)$.*

7. *Find* $\dfrac{dy}{dx}$ *at the given point:*

a) $\ln y - x = 0$ *at* $(1, e)$

b) $x \ln y + xy = 2$ *at* $(2, 1)$

8. *Find the equation of the tangent line and of the normal line to the curve $y = \sin x$ at the point (0, 0).*

9. *Find the equation of the tangent line and of the normal line to the curve $y = \cos x$ at the point $\left(\frac{\pi}{3}, \frac{1}{2}\right)$.*

10. *If $\tan(xy) = x$, find $\frac{dy}{dx}$ at the point $\left(1, \frac{\pi}{4}\right)$.*

11. *Find the tangent approximation to the function* $f(x) = \sqrt{x}$ *at x = 100 and use it to approximate the square root of 102.*

12. *Use Newton's Method to solve the equation* $x^2 - 2 = 0$ *using x-initial = 1*

EXTENDED QUESTIONS

Expand each logarithm.

1) $\ln \left(\frac{8^5}{7}\right)^4$

2) $\ln \left(c\sqrt{a \cdot b}\right)$

3) $\ln \left(uv^6\right)^5$

4) $\ln \left(x \cdot y \cdot z^6\right)$

Condense each expression to a single logarithm.

5) $25\ln 5 - 5\ln 11$

6) $5\ln x + 6\ln y$

7) $\frac{\ln 5}{2} + \frac{\ln 6}{2} + \frac{\ln 7}{2}$

8) $20\ln a - 4\ln b$

Solve each equation.Round your final answer to the nearest thousandth.

13) $\ln (7 - p) = \ln (-5p - 1)$

14) $\ln -2x = \ln (3x + 10)$

15) $\ln 8 - \ln (x + 4) = 1$

16) $\ln (x + 1) - \ln x = 5$

17) $\ln(x+8) - \ln 7 = 3$

18) $\ln 10 + \ln(5x-2) = 3$

19) $\ln(-4x-4) - \ln 3 = 3$

20) $\ln 3 + \ln(2x^2+4) = \ln 12$

Solve each equation. Round your answers to the nearest thousandth.

23) $-8e^{-p} = -49$

24) $2e^{8x} = 45$

25) $e^{-4x} + 8 = 35$

26) $e^{3r} + 4 = 59$

27) $3e^{-b} = 56$

28) $-9.3e^{10b} = -67$

29) $4e^{r+5} = 29$

30) $6e^{-10r} = 63$

5.2 - <u>Optimization</u>

This the process of finding the greatest (maximum optimal solution) or least value of a function (the minimum optimal solution) of a problem given some constraint.

This section is generally one of the more difficult for students taking a Calculus course. One of the main reasons for this is that a subtle change of wording can completely change the problem. There is also the problem of identifying the quantity that we'll be optimizing and the quantity that is the constraint and writing down equations for each.

The first step in all of these problems should be to very carefully read the problem so that you can create a suitable picture. Once you've done that the next step is to identify the quantity to be optimized and the constraint.

Example 1:

An open field is bounded by a lake with a straight shoreline. A rectangular enclosure is constructed using 800 m of fencing along three sides. What dimensions will maximum the enclosed area and what is the maximum area?

STEPS:

Example 2:

A rectangular poster board which is to contain a 9 m^2 print. The margins must be 3 m on each side and 3 m on the top and bottom. What dimensions will minimize the amount of material used?

Example 3:

A rectangular storage container with an open top need to have a volume of 10 m^3. The length of its base is twice the width. Material for the sides costs $6 per square meter. Material for the base costs $10 per square meter. Find the cost of material for the cheapest container?

5.2 - PRACTICE QUESTIONS

1. *An open field is bounded by a lake with a straight shoreline. A rectangular enclosure is to be constructed by using 500 m of fencing along three sides and the lake as a natural boundary on the fourth side. What dimensions will maximize the enclosed area and what is the maximum area?*

2. *Two farmers have 800 m of fencing. They wish to form a rectangular enclosure and then divide it into 3 sections by running two lengths of fence parallel to one side. What should the dimensions of the enclosure be in order to maximize the enclosed area?*

3. *A piece of wire 24 in long is used to form a square and a rectangle whose length is three times its width. Determine their minimum combined area.*

4. Find the dimensions of the rectangle of largest area whose base in on the x-axis and the upper two vertices lie on the parabola $y = 12 - x^2$. What is the maximum area?

5. An open box by cutting squares of equal size from the corner of a 24 cm by 15 cm piece of sheet metal and folding up the sides. Determine the size of the cut-out that maximizes the volume of the box.

6. An open box from a 12 in by 12 in piece of cardboard by cutting away squares of equal size from the other four corners and folding up the sides. Determine the size of the cut-out that maximizes the volume of the box.

7. *A rectangular poster which is to contain 50 cm^2 of print, must have margins of 2 cm on each side and 4 cm on the top and bottom. What dimensions will minimize the amount of material used?*

8. *Construct a closed rectangular box with square base which has a surface area of 150 cm^2. What is the maximum possible volume of such a box?*

9. *An open rectangular box with a base twice as long as it's wide. If its volume must be 972 cm^3, what dimensions will minimize the amount of material used in its construction?*

EXTENDED QUESTIONS

1. **Dimensions of a box**. *A closed 3-dimensional box is to be constructed in such a way that its volume is 4500 cm^3. It is also specified that the length of the base is 3 times the width of the base. Determine the dimensions of the box which satisfy these conditions and have the minimum possible surface area.*

2. **The largest garden**. *You are building a fence to completely enclose part of your backyard for a vegetable garden. You have already purchased material for a fence of a length 100 ft. What is the largest rectangular area that this fence can enclose?*

3. **Two gardens**. A fence of length 100 ft is to be enclose two gardens. One garden is to have a circular shape, and the other to be square. Determine how the fence should be cut so that the sum of the areas inside both gardens is as large as possible.

4. **Dimensions of open box.** A rectangular piece of cardboard with dimension 12 cm by 24 cm is to be made into an open box (no lid) by cutting out squares from the corners and then turning up the sides. Find the size of the squares that should be cut out if the volume of the box is to be a maximum.

5. **Cost with Fixed Area.** A fence must be built in a large field to enclose a rectangular area of 25,600 m^2. One side of the area is bounded by an existing fence (no fence needed there). Material for the fence cost \$3 per meter for the ends and \$1.50 per meter for the side opposite the existing fence. Find the cost of the least expensive fencing.

6. **Cost with Fixed Area.** A fence must be built to enclose a rectangular area of 20,000 ft^2. Fencing material costs \$2.50 per foot for the two sides facing north and south and \$3.20 per foot for the other two sides. Find the cost of the least expensive fence.

7. **Packaging Cost.** A closed box with a square base is to have a volume of 16,000 cm^3. The material for the top and bottom of the box costs \$3 per square centimeter, while the material for the sides costs \$1.50 per square centimeter. Find the dimension s of the box that will lead to the minimum total cost. What is the minimum total cost?

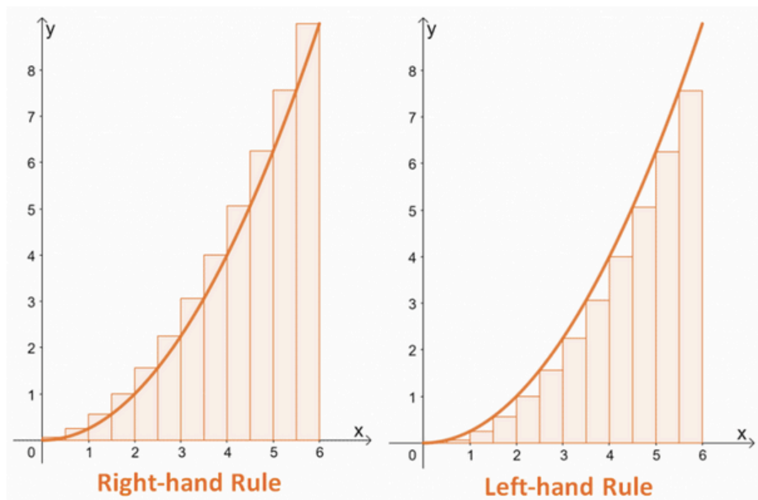
PROJECT – see your teacher to get > Hydro Project

5.3 – Riemann’s Sum

In this lesson, we will learn how to approximate the area under the curve using rectangles. This method is called "Riemann Sum". The method involves finding the length of each sub-interval (Δx), and finding the points of interest, finding the y values of each point of interest, and then use the find the area of each rectangle to sum them up. There are 3 methods in using the Riemann Sum. First is the "Right Riemann Sum", second is the "Left Riemann Sum", and third is the "Trapezoidal Riemann Sum"

This method is similar to definite integrals we did earlier. Riemann’s Sum is expressed:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \qquad \Delta x = \frac{b-a}{n} \qquad and \qquad x_i = a + i\Delta x$$



In the **right Riemann sum**, we construct the rectangles so that the **curve passes through the top-right right corners**.

Similarly, for the **left Riemann sum**, we construct the rectangles so that the **curve passes through the top-left corners.**

“OVERESTIMATE” “UNDERESTIMATE” ??

Example:

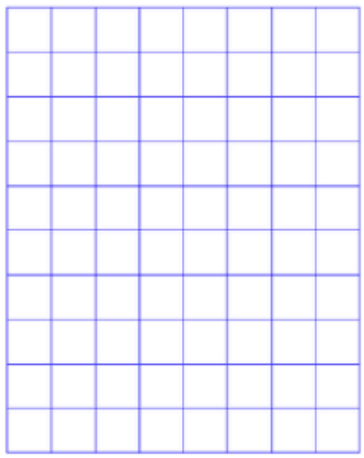
Let $f(x) = x^3$ from $[0, 3]$ using equal widths, with 3 rectangles.

5.3 - PRACTICE QUESTIONS

The rate that people are entering a local office is given below in people/hour. Use the table to answer questions 1-3.

Time (hours)	0	1	2	3	4
r ' (t) ppl/hr	12	7	3	5	8

Sketch and label a graph:



1. Use a left Riemann sum with 4 subintervals to approximate the total number of people entering the office over the interval $0 \le t \le 4$.

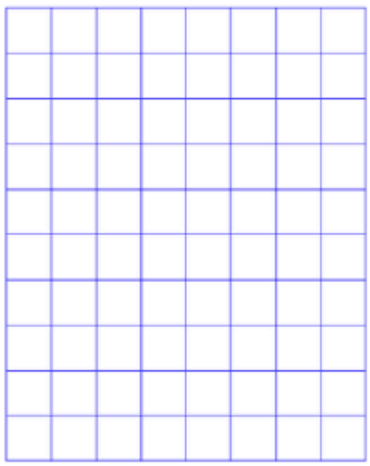
2. Use a right Riemann sum with 4 subintervals to approximate the total number of people entering the office over the interval $0 \le t \le 4$.

3. Use a trapezoidal approximation with 4 subintervals to approximate the total number of people entering the office over the interval $0 \le t \le 4$.

Gasoline is being pumped into a car. The rate that the gas is being pumped is given in the table below at selected times (seconds). Use the table to answer questions 4-6.

Time (sec)	0	4	8	12	16	20	24
$g'(t)$ gal/sec	0	.34	.42	.56	.45	.34	.22

Sketch and label a graph:



4. Use a left Riemann Sum with 3 subintervals to approximate the total gallons of gasoline pumped in the car over the 24 seconds.

5. Use a right Riemann Sum with 3 subintervals to approximate the total gallons of gasoline pumped in the car over the 24 seconds.

6. Use a trapezoidal Riemann Sum approximation with 3 subintervals to approximate the total gallons of gasoline pumped in the car over the 24 seconds.

PROJECT – see your teacher to get > Creek Project

5.4 – Mean Value, Rolle's & Intermediate Value Theorems

Mean Value Theorem

Suppose $f(x)$ is a function that satisfies both of the following.

1. $f(x)$ is continuous on the closed interval $[a, b]$.
2. $f(x)$ is differentiable on the open interval (a, b).

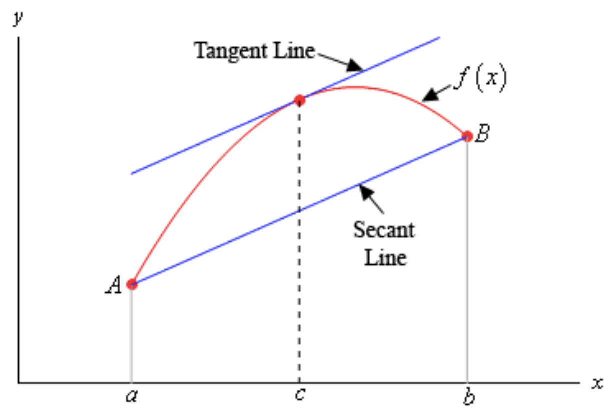
Then there is a number c such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or

$$f(b) - f(a) = f'(c)(b - a)$$

What the Mean Value Theorem tells us is that these two slopes must be equal or in other words the secant line connecting A and B and the tangent line at $x = c$ must be parallel. We can see this in the following sketch.



Examples:

a) Show how the **MEAN VALUE THEOREM** applies to the given function over the given interval:

$f(x) = x^2 - 4x \qquad 2 \le x \le 6$

b) Determine all the numbers c which satisfy the conclusions of the Mean Value Theorem for the following function: $f(x) = x^3 + 2x^2 - x$; on $[-1, 2]$

TRY:
Use the MVT to determine all the numbers c for the following function:
$f(x) = x^3 + 4x$, $[-1, 1]$

Rolle's Theorem

Rolle's Theorem is a special case of the Mean Value Theorem.

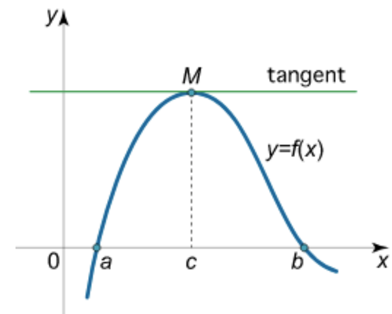
Suppose $f(x)$ is a function that satisfies all of the following.

1. $f(x)$ is continuous on the closed interval $[a, b]$.
2. $f(x)$ is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c such that $a < c < b$ and $f'(c) = 0$.

Or, in other words $f(x)$ has a critical point in (a, b).

There is a point c on the interval (a, b) where the tangent to the graph of the function is horizontal.



Example 1: $f(x) = x^2 - 3x + 2 \qquad [0, 3]$

Example 2: $f(x) = x\sqrt{4 - x} \qquad [0, 4]$

TRY:
Use Rolle's Theorem to show that the function has a horizontal tangent line in the interval [0, 2]

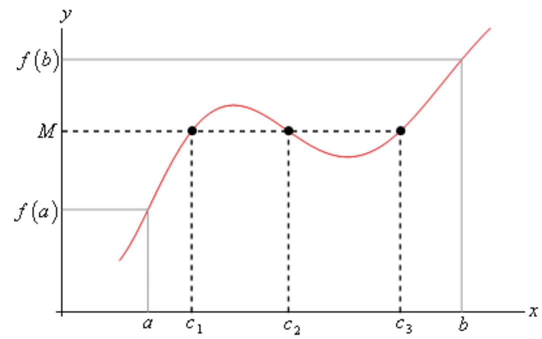
$f(x) = x^2 - 2x$

Intermediate Value Theorem

Suppose that $f(x)$ is continuous on $[a, b]$ and let M be any number between $f(a)$ and $f(b)$. Then there exists a number c such that,

1. $a < c < b$
2. $f(c) = M$

All the Intermediate Value Theorem is really saying is that a continuous function will take on all values between $f(a)$ and $f(b)$. Below is a graph of a continuous function that illustrates the Intermediate Value Theorem.



It's also important to note that the Intermediate Value Theorem only says that the function will take on the value of M somewhere between a and b. It doesn't say just what that value will be. It only says that it exists.

A nice use of the Intermediate Value Theorem is to prove the existence of roots of equations as the following example shows.

Ex. 1: Show that $p(x) = 2x^3 - x^2 - 10x + 5$ has a root somewhere in the interval $[-1,2]$.

Ex. 2: Use the IVT to show that there is a number c where $f(c) = 19$ on the interval $[1, 4]$ given the function $f(x) = 2x^2 - 3x + 5$. Find the value of c.

TRY 1:
Use the IVT to show that $f(x) = x^4 - 7x^2 + 10$ has a root somewhere in the interval $[0, 2]$

TRY 2:
Use the IVT to show that there is a number c where $f(c) = 9$ on the interval $[0, 3]$ given the function $f(x) = 2x^2 - 3x + 7$. Find the value of c.

5.4 - PRACTICE QUESTIONS

*1, Show how the **MEAN VALUE THEOREM** applies to the given function over the given interval:*

a) $f(x) = x^2, \quad 2 \leq x \leq 6$

b) $g(x) = x^2 + 3x + 5, \quad 1 \leq x \leq 4$

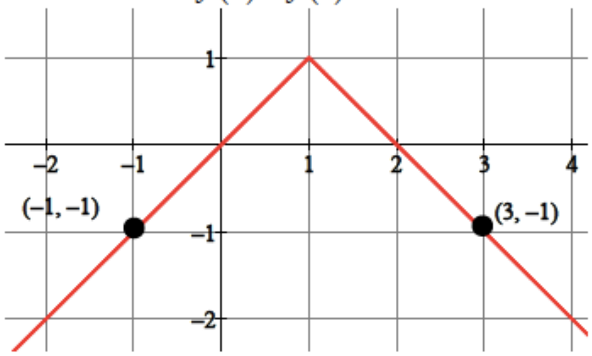
*2. Show how the **INTERMEDIATE VALUE THEOREM** applies to the given function over the given interval:*

a) $f(x) = x^2, \quad 2 \leq x \leq 8$ *and* $f(c) = 36$

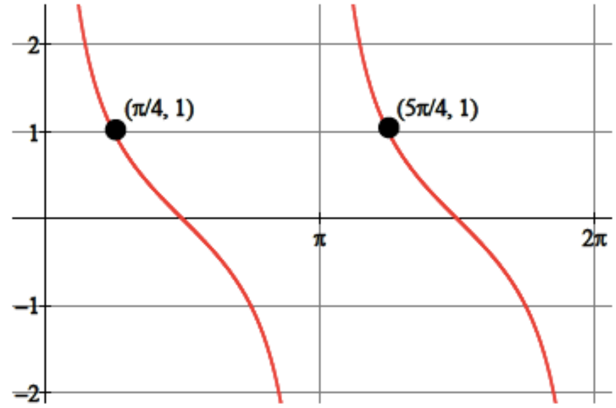
3. Use Rolle's Theorem to show that the function has a horizontal tangent in the interval [-1, 3] $\quad f(x) = x^2 - 2x + 1$

EXTENDED QUESTIONS

In problems 1 and 2, state why Rolle's Theorem does not apply to the function even though there exist a and b such that $f(a) = f(b)$.

1. 

$f(x) = 1 - |x - 1|$

2. 

$f(x) = \cot x$

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3. Determine whether the Mean Value Theorem (MVT) applies to the function $f(x) = 3x^2 - x$ on the interval $[-1, 2]$. If it applies, find all the value(s) of c guaranteed by the MVT for the indicated interval.

---

4. Determine whether the MVT applies to the function $f(x) = \dfrac{x+1}{x}$ on the interval $[-2, 3]$. If it applies, find all the value(s) of c guaranteed by the MVT for the indicated interval.

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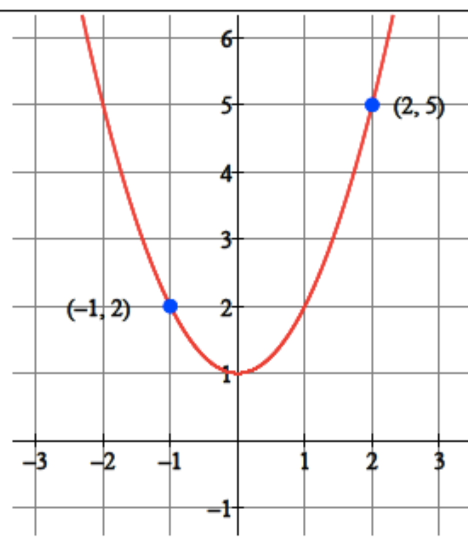
5. Consider the graph of the function $g(x) = x^2 + 1$ shown to the right.
   a) On the drawing provided, draw the secant line through the points $(-1, 2)$ and $(2, 5)$.
   b) Since g is both continuous and differentiable, the MVT guarantees the existence of a tangent line(s) to the graph parallel to the secant line. Sketch such line(s) on the drawing.
   c) Use your sketch from part (b) to visually estimate the x-coordinate at the point of tangency.
   d) That x-coordinate at the point of tangency is the value of c promised by the MVT. Verify your answer to part (c) by using the conclusion of the MVT on the interval $[-1, 2]$ to find c.



6. Given $h(x)=x^{2/3}$, explain why the hypothesis of the MVT are met on $[0, 8]$ but are not met on $[-1, 8]$.

Some of the following questions require using the IVT and MVT "backwards." This means that a fact is stated and you need to identify what theorem was used to guarantee that fact. You might want to read again the conclusions for the IVT and MVT before attempting these problems!

13. Given $h(x)=x^3+x-1$ on the interval $[0, 2]$, will there be a value p such that $0<p<2$ and $h'(p)=5$? Justify your answer. If your answer is yes, find p.

14. Given $g(x)=x^3-x^2+x$ on the interval $[1, 3]$, will there be a value r such that $1<r<3$ and $g(r)=11$? Justify your answer. If your answer is yes, use your calculator to find r.

You may use a calculator for this problem.

15. The height of an object t seconds after it is dropped from a height of 500 meters is $h(t)=-4.9t^2+500$.
   a) Find the average velocity of the object during the first three seconds. Remember: average velocity is equal to change in position divided by change in time.
   b) Show that at some time during the first three seconds of fall the instantaneous velocity must equal the average velocity you found in part (a). Then, find that time.

5.5 - Improper Integrals: Infinite Limits of Integration

Improper Integrals

To compute improper integrals, we use the concept of limits along with the Fundamental Theorem of Calculus (FTC).

Definition 2.52. Improper Integrals — One Infinite Limit of Integration. If $f(x)$ is continuous on $[a, \infty)$, then the improper integral of f over $[a, \infty)$ is

$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx.$$

If $f(x)$ is continuous on $(-\infty, b]$, then the improper integral of f over $(-\infty, b]$ is

$$\int_{-\infty}^b f(x)\,dx = \lim_{R\to-\infty} \int_R^b f(x)\,dx.$$

Since we are dealing with limits, we are interested in convergence and divergence of the improper integral. If the limit exists and is a finite number, we say the improper integral ***converges***. Otherwise, we say the improper integral ***diverges***, which we capture in the following definition.

Definition 2.53. Convergence and Divergence. If the limit exists and is a finite number, we say the improper integral ***converges***.

If the limit is $\pm\infty$ or does not exist, we say the improper integral ***diverges***.

Examples:

$a)\ \displaystyle\int_1^\infty \frac{1}{x^4}\,dx$

$b)\ \displaystyle\int_1^\infty 2x^3\,dx$

Exact Arc Length

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx$$

Example 1:

Find the exact arc length of the given function between the given values of *x*:

$f(x) = 3x$ from $x = 2$ to $x = 5$

Example 2:

Determine the arc length of $f(x) = 1 + 2x^{\frac{3}{2}}$ between $[0, 1]$, to 2 decimal places.

Partial Derivatives

Example: Given $\boldsymbol{f}$ is a function of x and y, find the partial derivative of $\boldsymbol{f}$ with respect to x and the partial derivative of $\boldsymbol{f}$ with respect to y of the following:

$$\boldsymbol{f}(x, y) = x^4 + 2x^3y^2$$

Double Integration

Example: Find the exact value of the following multiple integrals:

$$\int_0^1 \int_2^4 2x \; dx \, dy$$

Differential Equations or Separable Equations

Example1: Find all solutions of each differential equations below:

a) $\frac{dy}{dx} = x - 4$

b) $(x^2 + 1)\frac{dy}{dx} = xy$

Example2: Find all solution of the differential equation satisfying the given initial condition:

$$\frac{dy}{dx} = 3x^2 + 6, \quad y(1) = 9$$

5.5 - PRACTICE QUESTIONS

*1. Evaluate each of the following **IMPROPER INTEGRALS** if possible:*

a) $\int_{1}^{\infty} \frac{1}{x^2}\, dx$

b) $\int_{-\infty}^{0} 2e^x\, dx$

c) $\int_{0}^{1} \frac{1}{\sqrt{x}}\, dx$

d) $\int_{2}^{\infty} \frac{1}{x}\, dx$

2. Find the exact arc length of the given function between the given values of x:

a) $f(x) = 2x$, *from* $x = 4$ *to* $x = 8$

b) $g(x) = \frac{2}{3}x^{\frac{3}{2}}$, *from* $x = 1$ *to* $x = 3$

*3. Given **f** is a function of x and y find the partial derivative of **f** with respect to x and the partial derivative of **f** with respect to y of each of the following:*

a) $f(x, y) = 4x^3 - 3x^2y^2$

b) $f(x, y) = x^4 \ln y + y$

4. Find the exact value of each of the following multiple integrals:

a) $\int_0^1 \int_2^4 x \; dx\,dy$

b) $\int_b^a \int_{-1}^5 2x \; dx\,dy$

c) $\int_1^3 \int_{-1}^1 y^2 \; dy\,dx$

d) $\int_{-1}^1 \int_{-2}^2 2x + 2y \; dy\,dx$

5. Find all solutions of each ***DIFFERENTIAL EQUATION*** *below:*

a) $\dfrac{dy}{dx} = 2x + 3$ *b)* $\dfrac{dy}{dt} = 4t^2 + 5$ *c)* $\dfrac{dv}{dt} = 3\cos 2t - 4$

6. Find the solution of each of the ***SEPARABLE EQUATION*** *satisfying the given initial condition:*

a) $\dfrac{dy}{dx} = 3x^2 + 6, \quad y(1) = 9$

b) $\dfrac{dy}{dt} = (3 - 2t)^5, \quad y(2) = 1$

5.6 - Imaginary Numbers or Complex Numbers

Imaginary numbers, also called complex numbers, are used in real-life applications, **such as electricity**, as well as quadratic equations. In quadratic planes, imaginary numbers show up in equations that don't touch the x axis. Imaginary numbers become particularly useful in advanced calculus.

Example: Simplify a) $\sqrt{-9}$ b) i^5

Complex Numbers

Examples:

a) Find the absolute value of the following complex numbers: $|4+2i|$

b) Simplify each:

$i)$ $(2+6i)+(4-8i)$ $ii)$ $(2+5i)(2-5i)$

$iii)$ $\frac{4-6i}{2+i}$

c) Solve the following equation over the complex numbers: $x^2+9=0$

5.6 - PRACTICE QUESTIONS

1. Simplify:

a) $\sqrt{-9}$ b) $\sqrt{-64}$ c) $\sqrt{-20}$ d) $\sqrt{-80}$ e) $\sqrt{-2}$

f) i^5 g) i^6 h) i^{83} i) i^{202} j) i^{484}

2. Simplify:

a) $(2 + 6i) + (4 - 8i)$ *b)* $(3 - 5i) - (6 - 9i)$ *c)* $(2 + 5i)(3 - 4i)$

d) $(2 + 5i)(2 - 5i)$ *e)* $\dfrac{6}{5i}$

h) $\dfrac{4 - 6i}{2 + i}$ *i)* $\dfrac{8 - 2i}{8 + 2i}$

3. Solve each of the following equations over the COMPLEX NUMBERS:

a) $x^2 + 9 = 0$

b) $3x^2 + 10 = 2x^2 + 8$

c) $x^2 + 2x + 2 = 0$

d) $2x^2 = 2x - 3$

4. Find the following integrals over the COMPLEX NUMBERS:

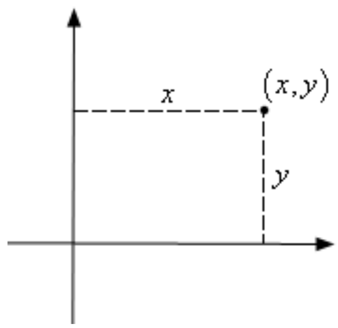
a) $\int_{-2i}^{4i} 2x \, dx$

b) $\int_{-2i}^{i} 4x^3 \, dx$

5.7 - Polar Coordinates

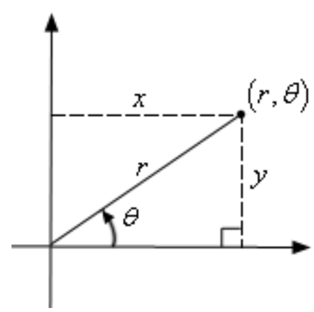
Up to this point in math we've dealt exclusively with the Cartesian (or Rectangular, or *x*-*y*) coordinate system. However, as we will see, this is not always the easiest coordinate system to work in. So, in this section we will start looking at the polar coordinate system.

Coordinate systems are really nothing more than a way to define a point in space. For instance in the Cartesian coordinate system at point is given the coordinates **(x, y)** and we use this to define the point by starting at the origin and then moving ***x*** units horizontally followed by ***y*** units vertically. This is shown in the sketch below.



This is not, however, the only way to define a point in two-dimensional space. Instead of moving vertically and horizontally from the origin to get to the point we could instead go straight out of the origin until we hit the point and then determine the angle this line makes with the positive *x*-axis. We could then use the distance of the point from the origin and the amount we needed to rotate from the positive *x*-axis as the coordinates of the point. This is shown in the sketch below.

Coordinates in this form are called **polar coordinates**.



Basics

$$r^2 = x^2 + y^2$$

$$r = \sqrt{x^2 + y^2}$$

$$Sin\ \theta = \frac{y}{r} \qquad Cos\ \theta = \frac{x}{r} \qquad Tan\ \theta = \frac{y}{x}$$

$$y = rSin\ \theta \qquad x = rCos\ \theta \qquad \theta = Tan^{-1}\left(\frac{y}{x}\right)$$

Convert Polar to Rectangular Coordinates

Example: $\left(2\sqrt{2}, \frac{3\pi}{4}\right)$

Convert Rectangular to Polar Coordinates

Example: $\left(-1, -\sqrt{3}\right)$

Find Rectangular Equation

Example 1: $r = 3$

Example 2: $r = 6sin\theta$

Example 3: $r = 4csc\theta$

Example 4: $r = 6sin\theta + 4cos\theta$

Example 5: $r = \frac{2}{5cos\theta+sin\theta}$

Example 6: $r = 2cos\theta tan\theta$

Find Polar Equation

Example 1: $x^2 + y^2 = 25$

Example 2: $x + 3y = 5$

Example 3: $x^2 + y^2 = 36$

Example 4: $(x + 4)^2 + (y - 1)^2 = 17$

5.7 - PRACTICE QUESTIONS

1. Give the relationship between the POLAR COORDINATES (r, θ) and the RECTANGULAR COORDINATES (x, y):

2. Give the rectangular coordinates of the point whose polar coordinates are given:

a) $\left(2, \frac{\pi}{6}\right)$ *b)* $\left(-4, \frac{\pi}{3}\right)$ *c) (2, 0)*

3. Give the polar coordinates of a point with the given rectangular coordinates:

a) (1, 0) *b) (1, 1)* *c)* $\left(\sqrt{3}, 1\right)$

4. *Find a rectangular equation equivalent to the given polar equation and describe the graph:*

a) $r = 2$

b) $r = a$

c) $\theta = \frac{\pi}{4}$

d) $r = 2\sin\theta$

e) $r = 4\cos\theta$

f) $r = \tan\theta\sec\theta$

5. *Change the given rectangular equation into an equivalent polar equation:*

a) $x^2 + y^2 = 16$

b) $x = 3$

c) $2x^2 + 2y^2 = 8$

d) $y = \sqrt{3}x$

e) $x + 2y = 3$

f) $x^2 - y^2 = 1$

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