PRE-CALCULUS 11

Seminar Notes

Learning Guides 5 & 6

QUADRATIC FUNCTIONS

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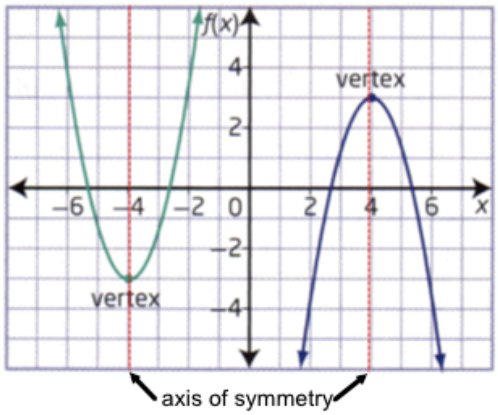
Topic 1 Quadratics

A quadratic is a function where the x value is squared. The simplest quadratic is $f(x) = x^2$



- The graph of a quadratic function is a parabola
- The lowest or highest point on the graph is the vertex
- The axis of symmetry divides the graph into mirror images and its' equation corresponds to the x-coordinate of the vertex

Quadratic Functions can be written in vertex form or standard form. Vertex form is useful for graphing.

vertex form

$$f(x) = a(x-p)^2 + q$$

stretch/shrink
opens up or down

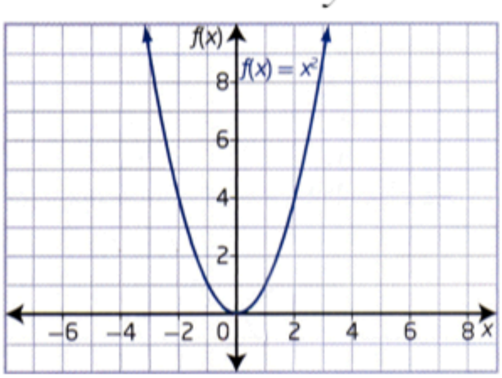
vertex (p, q)

standard form

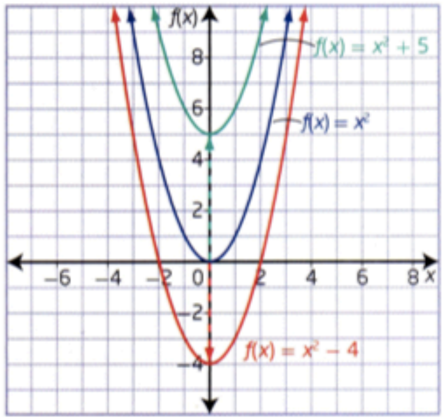
$$f(x) = ax^2 + bx + c$$

The effects of a, p & q on the function

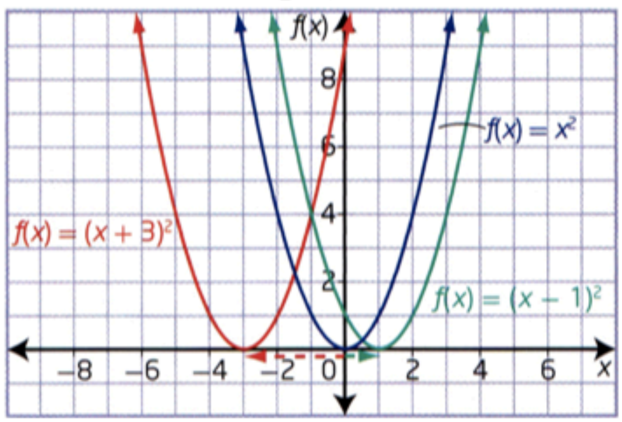
1. Basic Parabola $y = x^2$



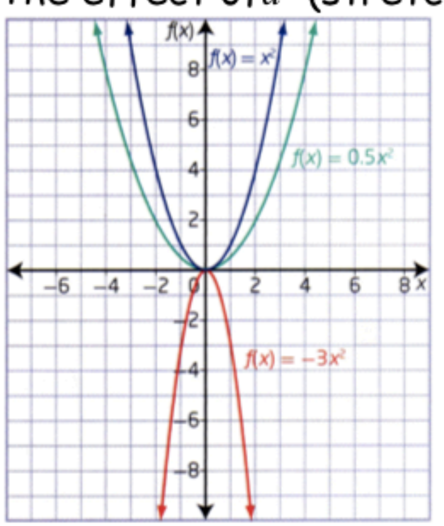
2. The effect of q (vertical shift) $y = x^2 + q$



3. The effect of p (horizontal shift) $y = (x - p)^2$



4. The effect of a (stretch & shrink) $y = ax^2$



The value of a tells you 2 things:

a) graph opens up or down

- $a > 0$ opens up
- $a < 0$ opens down

b) stretch or shrink

- $-1 < a < 1$ graph is wider than $y = x^2$
- $a > 1$ or $a < 1$ graph is more narrow than $y = x^2$

Example 1

Sketch Graphs of Quadratic Functions in Vertex Form

Example: Determine the following for a function:

- the vertex
- the domain and range
- the direction of the opening
- the equation of the axis of symmetry

Then, sketch the graph

$$y = 2(x+1)^2 - 3$$

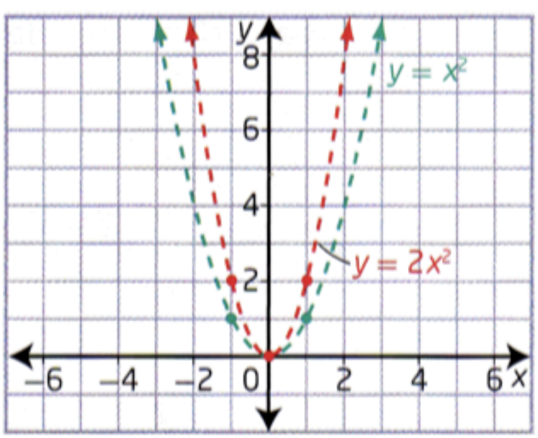
$a = 2$ $p = -1$ $q = -3$

vertex: (-1, -3)
opens: up (a > 0) and is narrower than $y = x^2$ (a > 1)
domain: $\{x | x \in R\}$ or x = all real numbers (for all parabolas)
range: $\{y | y \geq -3, y \in R\}$ or y ≥ -3 (q = -3)
axis of symmetry: x = -1 (p = -1)

Example 1 (cont.)

Method 1: Sketch using Transformations

1. Start with the graph of $y = x^2$
   - Use the points (0, 0), (1, 1), (-1, 1), (2, 4) and (-2, 4) to graph $y = x^2$



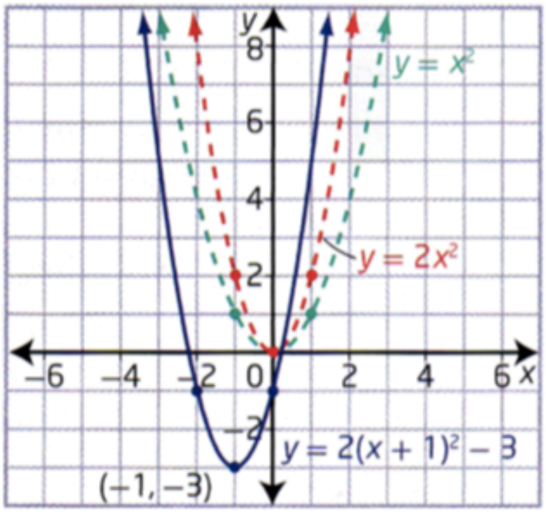
- Apply the change in width (a) first:

(0, 0) → (0, 0)
(1, 1) → (1, 2)
(-1, 1) → (-1, 2)
(2, 4) → (2, 8)
(-2, 4) → (-2, 8)

Example 1 (cont.)

Method 1: Sketch using Transformations

2. Translate the graph
   - Use the values of p and q to give the vertical and horizontal translation

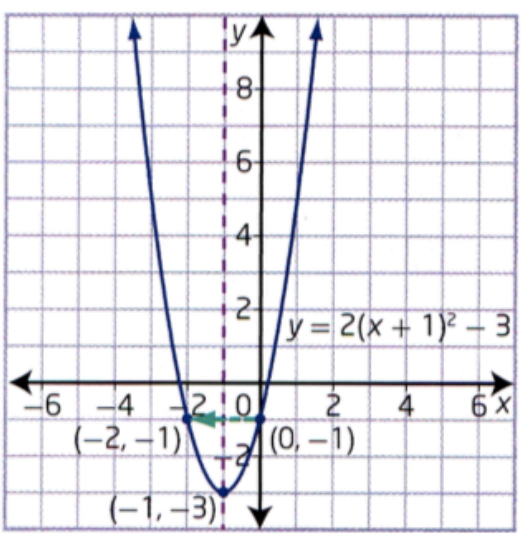


- $p = -1$, so the graph is translated one unit left
- $q = -2$, so the graph is translated two units down

Example 1 (cont.)

Method 2: Sketch using Points and Symmetry

- Plot the vertex, (-1, -3), and draw the axis of symmetry, x = -1
- determine the coordinates of at least 4 more points



a) Let $x = 0$

$$y = 2(0+1)^2 - 3$$

$$y = 2(1)^2 - 3$$

$$y = -1$$

The point is (0, -1) and there is a matching point across the axis of symmetry at (-2, -1)

Try: Determine the following for each function:

- the vertex
- the domain and range
- the direction of the opening
- the equation of the axis of symmetry

a) $y = \frac{1}{2}(x-2)^2 - 4$

Vertex:

D: R:

Direction Opening:

Equ. of Axis of Symm:

b) $y = -3(x+1)^2 + 3$

Vertex:

D: R:

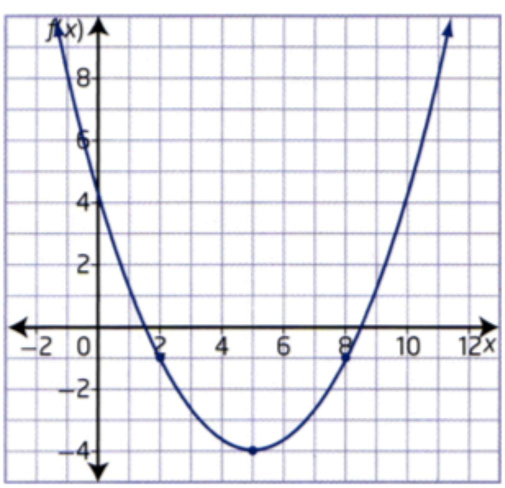
Direction Opening:

Equ. of Axis of Symm:

Example 2

Determine a Quadratic Function Given Its Graph

Example: Determine a quadratic function in vertex form for the following graph.



Use Points and Substitution

1. Use the coordinates of the vertex and one other point:
   vertex (5, -4) and P (2, -1)
2. Substitute 5 and -4 for p and q into the vertex form of the equation.

$$f(x) = a(x-p)^2 + q$$

$$f(x) = a(x-5)^2 + (-4)$$

$$f(x) = a(x-5)^2 - 4$$

Example 2 cont.

3. Solve the equation for a by substituting (2, -1) for x and y

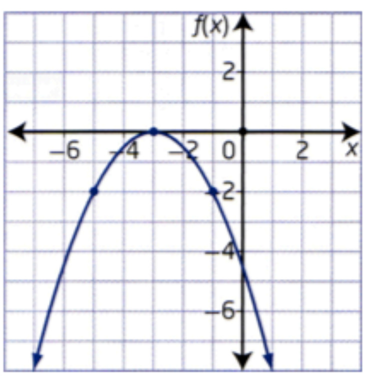
$$f(x) = a(x-5)^2 - 4$$
$$-1 = a(2-5)^2 - 4$$
$$-1 = a(-3)^2 - 4$$
$$-1 = a(-9) - 4$$
$$3 = 9a$$
$$\frac{1}{3} = a$$

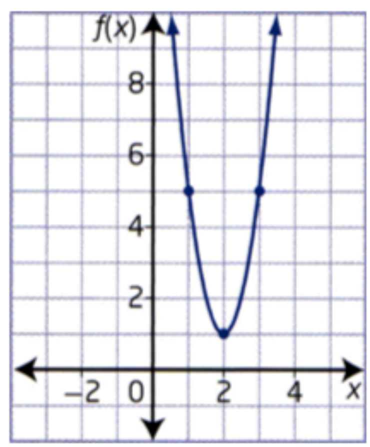
4. Rewrite equation with using a, p and q (but not x and y)

$$f(x) = a(x-p)^2 + q$$
$$f(x) = \frac{1}{3}(x-5)^2 - 4$$

Try: Determine a quadratic function in vertex form for the following graphs.

a)





Example 3

Determine the Number of x-intercepts Using a and q

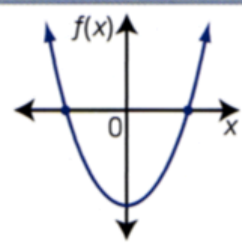
Example: Determine the number of x-intercepts for each quadratic function:

a) $f(x) = 0.8x^2 - 3$ b) $f(x) = 2(x-1)^2$ c) $f(x) = -3(x+2)^2 - 1$

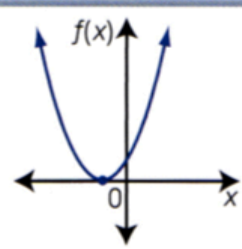
You need to know:

- the value of a to determine if the graph opens up or down
- the value of q to determine if the vertex is above, below or on the x-axis

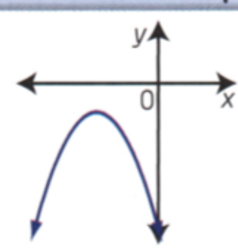
a) $f(x) = 0.8x^2 - 3$

Value of a	Value of q	Visualize the Graph	Number of x-Intercepts
$a > 0$ the graph opens upward	$q < 0$ the vertex is below the x-axis		2 crosses the x-axis *twice*, since it opens *upward* from a vertex *below* the x-axis

b) $f(x) = 2(x - 1)^2$

Value of a	Value of q	Visualize the Graph	Number of x-Intercepts
$a > 0$ the graph opens upward	$q = 0$ the vertex is on the x-axis		1 touches the x-axis *once*, since the vertex is *on* the x-axis

c) $f(x) = -3(x + 2)^2 - 1$

Value of a	Value of q	Visualize the Graph	Number of x-Intercepts
$a < 0$ the graph opens downward	$q < 0$ the vertex is below the x-axis		0 does not cross the x-axis, since it opens *down* from a vertex *below* the x-axis

Try: Determine the number of x-intercepts without graphing (use a and q)

a) $f(x) = 0.5x^2 - 7$

b) $f(x) = -2(x+1)^2$

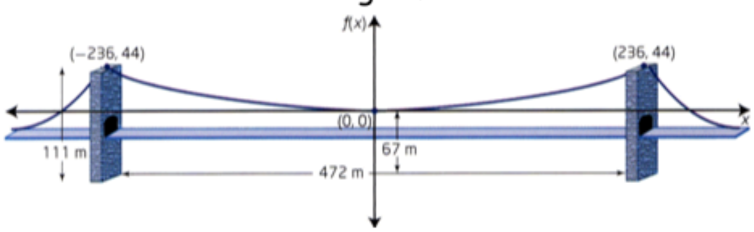
Example 4

Model Problems Using Quadratic Functions in Vertex Form

Example: The deck of a bridge is supported by 2 main cables attached to the tops of two towers. The cables are shaped like parabolas, with the lowest point approximately 67 m above the water. The towers are 111 m tall and 472 m apart.

a) Model the shape of the cables with a quadratic function in vertex form

1. Draw a labelled diagram



- place the vertex at the cables' low point & make it the origin
- put in the x and y-axes
- label the coordinates of the tops of the towers with respect to the vertex

2. Determine the form of the equation.
   - Since a and q are both 0, the function will have the form $f(x) = ax^2$

3. Determine the equation.
   - Substitute the coordinates of one the towers into $f(x) = ax^2$ and solve for a

$$f(x) = ax^2$$

$$44 = a(236)^2$$

$$44 = 55696a$$

$$\frac{44}{55696} = \frac{55696}{55696}a$$

$$\frac{11}{13924} = a$$

3. Re-write the equation with a in place

$$f(x) = \frac{11}{13924}x^2$$

Example 4 cont.

b) Determine the height above the surface of the water of a point on the cables that is 90 m horizontally from one of the towers.

1. Determine the distance of the point from the vertex.
   - A point 90 m from one of the towers is 236-90, or 146 m horizontally from the vertex

2. Use the equation for the function to determine $f(146)$

$$f(x) = \frac{11}{13924}x^2$$

$$f(146) = \frac{11}{13924}(146)^2$$

$$f(146) = \frac{11}{13924}(21316)$$

$$f(146) = 16.839\ldots$$

This is approximately 16.8 m above the low point in the cables which are 67 m above the water. The height above the water is 67 + 16.8 = 83.8 m

Try: A parabolic archway has a width of 280 cm and a height of 216 cm at its highest point.

a) Write a quadratic function in vertex form that models the shape of the archway

b) Determine the height of the archway at a point 50 cm from its outer edge.

Topic 2

Example 1

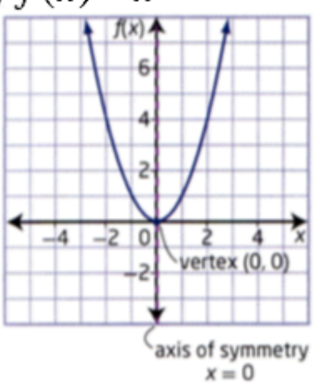
Quadratic Functions in Standard Form

Identify Characteristics of a Quadratic Function in Standard Form

Example: For each graph of a quadratic function, identify:

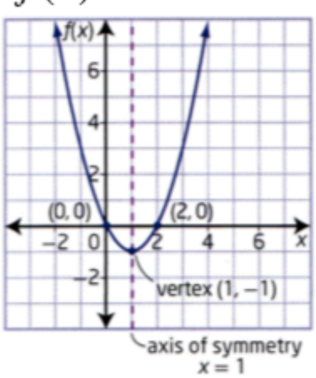
- the direction of the opening
- the coordinates of the vertex
- the maximum or minimum value
- the equation of the axis of symmetry
- the x-intercepts and the y-intercept
- the domain and range

a) $f(x) = x^2$



- opens upward
- vertex: (0, 0)
- minimum: y = 0 when x = 0
- axis of symmetry: x = 0
- y-intercept at (0, 0) and has a value of 0
- x-intercept at (0, 0) and has a value of 0
- domain: all real numbers or $\{x | x \in R\}$
- range: $y \geq 0$ or $\{y | y \geq 0, y \in R\}$

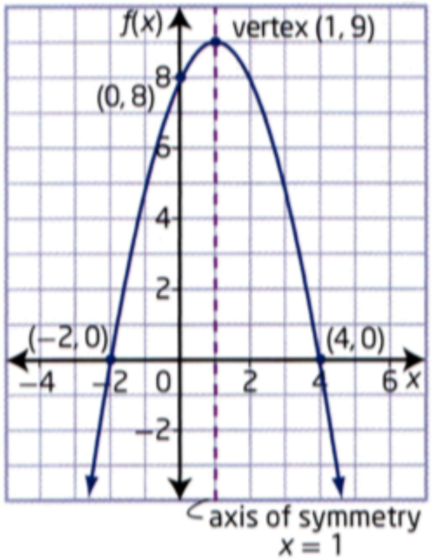
b) $f(x) = x^2 - 2x$



- opens upward
- vertex: (1, -1)
- minimum: y = -1 when x = 1
- axis of symmetry: x = 1
- y-intercept at (0, 0) and has a value of 0
- x-intercepts at (0, 0) & (2, 0) & have values of 0 & 2
- domain: all real numbers or $\{x | x \in R\}$
- range: $y \geq -1$ or $\{y | y \geq -1, y \in R\}$

Example 1 cont.

c) $f(x) = -x^2 + 2x + 8$

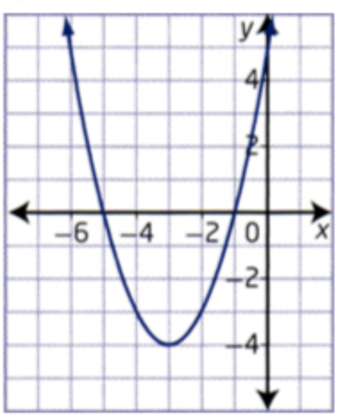


- opens downward
- vertex: (1, 9)
- maximum: $y = 9$ when $x = 1$
- axis of symmetry: $x = 1$
- y-intercept at (0, 8) and has a value of 8
- x-intercept at (-2, 0) & (4, 0) & have values of -2 & 4
- domain: all real numbers or $\{x | x \in R\}$
- range: $y \leq 9$ or $\{y | y \leq 9 | y \in R\}$

Try: For each graph of a quadratic function, identify:

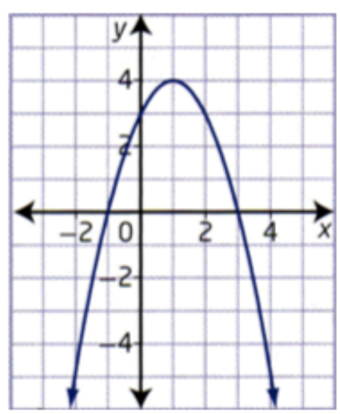
- the direction of the opening
- the coordinates of the vertex
- the maximum or minimum value
- the equation of the axis of symmetry
- the x-intercepts and the y-intercept
- the domain and range

a) $f(x) = x^2 + 6x + 5$



Dir. Opening:
Vertex:
Max/Min Value:
Equ. AoS:
x-int: y-int:
D: R:

b) $f(x) = -x^2 + 2x + 3$



Dir. Opening:
Vertex:
Max/Min Value:
Equ. AoS:
x-int: y-int:
D: R:

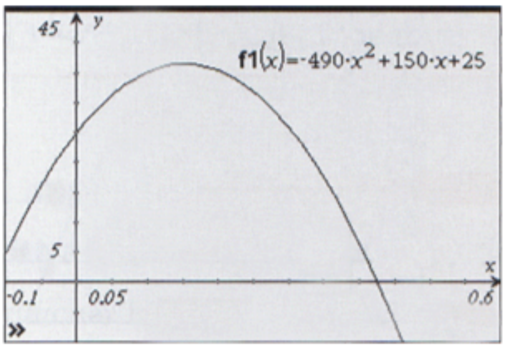
Example 2

Analysing a Quadratic Function

Example: A frog jumps into a pond. The height, h, in cm, of the frog above the water is a function of time, t, in seconds can be modeled by the function: $h(t) = -490t^2 + 150t + 25$

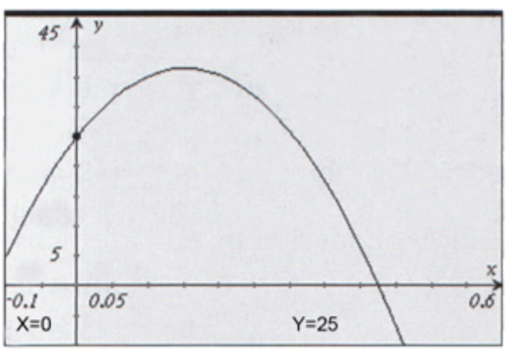
Answer the following:

a) Graph the function - use a graphing calculator



1. Press Y= and enter the function in Y_1= . Press GRAPH
2. You may have to adjust the size of the graph until the vertex and intercepts are visible. Press WINDOW and change Xmin, Xmax, Ymin and Ymax as necessary and press GRAPH

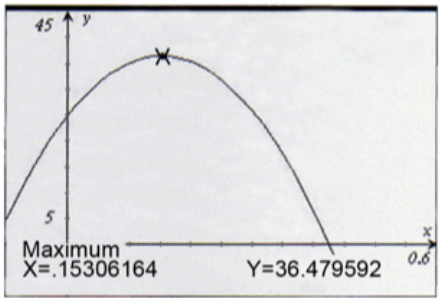
b) Find the y-intercept. What does it represent?



1. Press TRACE 0 ENTER
2. The y-intercept is (0, 25) or 25
   *note that this is the constant term in the equation
   $h(t) = -490t^2 + 150t + 25$
3. The y-intercept tells us that the height of the frog at the start of its jump was 25cm.

Example 2 cont.

c) What maximum height does the frog reach? When does it reach that height?

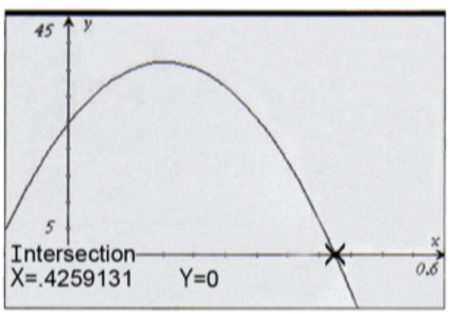


1. The vertex represents the time and height of the frog at its maximum point during the jump.

2. Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$ 4 (maximum) move the cursor to the left side of the maximum and press $\boxed{\text{ENTER}}$ then move the cursor to the right side of the maximum and press $\boxed{\text{ENTER}}$

3. The maximum occurs after about 0.2 s and the frog achieves a maximum height of about 36.5 cm

d) When does the frog hit the water?



1. The positive x-intercept represents the time when the height is 0 cm, or when the frog hits the water.

2. Press $\boxed{\text{Y=}}$ $\boxed{0}$ $\boxed{\text{GRAPH}}$

3. Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$ 5 (intersection) Make sure the cursor is on the right side of the maximum and press $\boxed{\text{ENTER}}$ three times.

4. The frog hits the water after approximately 0.2 s

e) What are the domain and range in this situation?

1. The domain is the set of all possible values for the independent variable (time).

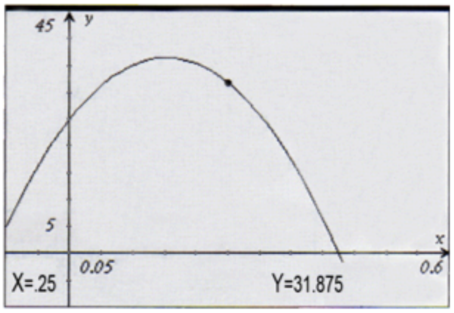
   $\{t \mid 0 \le t \le 0.4,\ t \in R\}$ Notice we start at 0 since negative values don't make sense for time in this case.

2. The range is the set of all possible values for the dependent variable (height).

   $\{h \mid 0 \le h \le 36.5,\ h \in R\}$ Notice we start at 0 since negative values don't make sense for height in this case.

Example 2 cont.

f) How high is the frog $0.25\ s$ after it jumps?



1. The height of the frog after 0.25 s is the h-coordinate when h is 0.25.

2. Press $\boxed{\text{TRACE}}$ 0.25 $\boxed{\text{ENTER}}$

3. The height of the frog is approximately 31.9 cm after 0.25 s

4. You can also calculate the height algebraically by substituting 0.25 for t in the function:

$$h(t) = -490t^2 + 150t + 25$$

$$h(0.25) = -490(0.25)^2 + 150(0.25) + 25$$

$$h(0.25) = -30.625 + 37.5 + 25$$

$$h(0.25) = 31.875$$

Try: A diver jumps from a 3m springboard with an initial velocity of 6.8 m/s. Her height, h, in metres, above the water t seconds after leaving the board can be modelled by the function

$$h(t) = -4.9t^2 + 6.8t + 3$$

a) Graph the function

b) What does the y-intercept represent?

c) What is her maximum height? When does she reach that height?

d) How long until she hits the water?

e) What domain and range are appropriate for this situation?

f) What is the height of the diver 0.6 s after leaving the board?

b)

c)

d)

e)

f)

LEARNING GUIDE 6

Topic 1

Example 1

Completing the Square

Example: Convert from Standard Form to Vertex Form.

$f(x) = x^2 + 6x + 5$

1. Group the 1st 2 terms

   $y = \left(x^2 + 6x\right) + 5$

2. Add and subtract the square of half the coefficient of the x-term (inside the brackets)

   $y = \left(x^2 + 6x + 9 - 9\right) + 5$

3. Group the trinomial

   $y = \left(x^2 + 6x + 9\right) - 9 + 5$

4. Factor and Simplify

   $y = (x+3)^2 - 9 + 5$

   $y = (x+3)^2 - 4$

Try: Convert each of the following to vertex from standard form:

a) $f(x) = x^2 + 16x + 20$

b) $f(x) = 3x^2 - 12x - 9$

Example 2

Convert to Vertex Form and Verify

Example: Convert the function $y = 4x^2 - 28x - 23$ to vertex form.

A good strategy for solving this type where you can't factor to get x^2 is use $x = \frac{-b}{2a}$

**just remember $x = p$ and $y = q$ in the vertex form: $y = a(x - p)^2 + q$

$a = 4$

$b = -28$ $\quad$ $x = \frac{-(-28)}{2(4)}$ $\quad$ $x = \frac{28}{8} = \frac{7}{2}$ $\therefore$ $p = \frac{7}{2}$

To find q, plug the x value you found above into the original equation $y = 4x^2 - 28x - 23$

$y = 4\left(\frac{7}{2}\right)^2 - 28\left(\frac{7}{2}\right) - 23 \quad y = -72 \;\therefore\; q = -72$

Now plug in the a, p and q into $y = a(x - p)^2 + q$

$$y = 4\left(x - \frac{7}{2}\right)^2 - 72$$

Try: Convert into vertex form.

a) $y = -3x^2 - 27x + 13$

Example 4

Write a Quadratic Model Function

Example: Last year photo sessions were \$10 and 400 sessions were booked. It is estimated that for every \$1 increase in price, 20 fewer sessions will be booked.

a) Write a function to model the situation

let n = the number of price increases
let R = the expected revenue
} Declare variables

$10 + 1n$ ← this years price
$400 - 20n$ ← number of sessions this year
} Write relationships

Revenue = (price)(number of sessions)

$R = (10 + n)(400 - 20n)$

$R = 4000 + 200n - 20n^2$

$R = -20n^2 + 200n + 4000$

} Write equation

b) Complete to find the maximum revenue and the price that brings that revenue

$R = -20n^2 + 200n + 4000$

$R = -20\left(n^2 + 10n\right) + 4000$

$R = -20\left(n^2 + 10n - 25 - 25\right) + 4000$

$R = -20\left[\left(n^2 + 10n - 25\right) - 25\right] + 4000$

$R = -20\left[(n - 5)^2 - 25\right] + 4000$

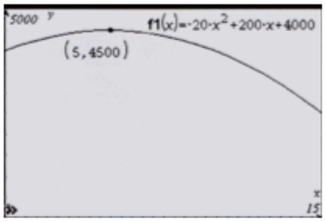
$R = -20(n - 5)^2 + 500 + 4000$

$R = -20(n - 5)^2 + 4500$

The vertex is (5, 4500). The maximum revenue will be \$4500 when n = 5 (when there are 5 price increases of \$1). So the price should go up \$5 to \$15.

Example 4 cont.

c) Verify the solution by graphing



1. Enter the function into y= and adjust the window. Then find the maximum like you did in learning guide 5
2. Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$ 4 (maximum) move the cursor to the left side of the maximum and press $\boxed{\text{ENTER}}$ then move the cursor to the right side of the maximum and press $\boxed{\text{ENTER}}$
3. The vertex is at (5, 4500). This verifies that the maximum revenue is $4500 with 5 price increases or a session fee of $15.

d) What assumptions were made in creating and using this model function?

It was assumed that:

· price affects revenue in a predictable way

Other factors could affect the revenue but were not considered, such as:

- advertising
- word of mouth
- ...

Try: A sporting goods store sells water bottles for $8. At this price they sell about 100 bottles per week. Research says that for every $2 increase in price, they can expect to sell 5 fewer bottles.

a) Write a quadratic to represent this situation

b) Find the maximum revenue and the selling price

c) Verify the solution

d) What were your assumptions?